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Monitoring parameter change for time series models with conditional heteroscedasticity

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HIGHLIGHTS

- We consider the sequential monitoring method based on the CUSUM of score functions.
- Our findings in the simulation study support the validity of our monitoring method.
- The proposed method is recommendable particularly when one aims to detect a change for one specific parameter in GARCH-type models.

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ABSTRACT

This paper studies the monitoring procedure to detect a parameter change in GARCH-type models based on the cumulative sum (CUSUM) of score functions as in Gombay and Serban (2009). For illustration, a simulation study is carried out for asymmetric GARCH models.

1. Introduction

In this paper, we study the CUSUM monitoring procedure, designed to sequentially detect a change of parameter vector components in time series models with heteroscedasticity. Since (Page, 1954), the CUSUM test has been a useful device for this purpose. Gombay (2003) and Gombay and Serban (2009) used the CUSUM approach based on the score vectors for independent observations, and later, extended it to autoregressive processes. Unlike the likelihood-ratio based monitoring procedure, they measure the type 1 error with probability rather than the average run length (ARL). The sequential monitoring method of Gombay and Serban (2009) has merits to attain lower false alarm rate. In our study, we focus on the monitoring process for generalized autoregressive conditional heteroscedastic (GARCH) type processes. Since the seminal paper of Engle (1982), GARCH processes have been playing a central role in modeling volatile time series. Among them, asymmetric GARCH (AGARCH) models are well known to properly describe the properties of the financial time series. Our CUSUM monitoring process is based on the asymptotic property of the partial sum of score vectors obtained from Gaussian quasi maximum likelihood (QML) functions (Francq and Zakoïan, 2004). Our method differs from that of Gombay and Serban (2009) in that the asymptotic result does not rely on the convergence rate of the parameter estimators induced from the law of iterated logarithm.

This paper is organized as follows. Section 2 states the properties of the QMLE in GARCH-type processes. Section 3 introduces the CUSUM monitoring procedure and provides the asymptotic behavior of the stopping rule. Section 4 evaluates its performance in AGARCH(1,1) processes. Section 5 provides the proof.

2. Conditionally heteroscedastic time series models

We consider the conditionally heteroscedastic model:

$$\begin{aligned} X_t &= \sigma_t Z_t, \\ \sigma_t^2 &= g_\theta(X_{t-1}, \ldots, X_{t-p}, \sigma_{t-1}^2, \ldots, \sigma_{t-q}^2) \end{aligned} \tag{1}$$

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for $t \in \mathbb{Z}$, where $\{g_\theta : \theta \in \Theta\}$ denotes a parametric family of nonnegative functions on $\mathbb{R}^p \times [0,\infty)^q$, θ belongs to a compact subset K of $\mathbb{R}^d$, (Z_t) are i.i.d. random variables with $\mathrm{E}Z_t = 0$, $Var(Z_t) = 1$, and $\sigma_t \geq 0$ is $\mathcal{F}_{t-1}$-measurable. Here $\mathcal{F}_t = \sigma(Z_k; k \leq t)$, $t \in \mathbb{Z}$. One can use notation $\mathbf{X}_t = (X_t, \ldots, X_{t-p+1})^T$ and $\boldsymbol{\sigma}_t^2 = (\sigma_t^2, \ldots, \sigma_{t-q+1}^2)^T$ to write $\sigma_t^2 = g_\theta(\mathbf{X}_t, \boldsymbol{\sigma}_t^2)$. In this study, we focus on the case of $p = 1$, $q = 1$.

Straumann and Mikosch (2006) embeded this model into a stochastic recurrence equation (SRE) and provided sufficient conditions for the stationarity, consistency, and asymptotic normality. More precisely, we observe σ_t^2 as a solution of the SRE: $s_{t+1} = \psi_t(s_t)$, $t \in \mathbb{Z}$, where $\psi_t(s) = g(s^{1/2}Z_t, s)$. The following is due to Proposition 3.1 of Straumann and Mikosch (2006).

Proposition 1. *For an arbitrary $\varsigma_0^2 \in [0,\infty)$, suppose that the following conditions hold for the stationary ergodic sequence (ψ_t):*
(S1) $\mathrm{E}[\log^+|\psi_0(\varsigma_0^2)|] < \infty$
(S2) $\mathrm{E}[\log^+ \Lambda(\psi_0)] < \infty$ *and for some integer* $r \geq 1$, $\mathrm{E}[\log \Lambda(\psi_0^{(r)})] = \mathrm{E}[\log \Lambda(\psi_0 \times \cdots \times \psi_{-r+1})] < 0$, *where* $\Lambda(\psi) = \sup_{x,y\in E, x\neq y}\left(\frac{d(\psi(x),\psi(y))}{d(x,y)}\right)$ *for any map* $\psi : E \to E$.

Then, the SRE admits a unique stationary ergodic solution σ_t^2. In particular, we can write

$$\sigma_t^2 = \lim_{m\to\infty} \psi_{t-1} \circ \cdots \circ \psi_{t-m}(\varsigma_0^2),\ t \in \mathbb{Z}, \tag{2}$$

where the latter limit is irrespective of ς_0^2.

Let (X_t, σ_t^2) be the stationary ergodic solution of model (1) with true parameter $\theta = \theta_0$. With initial value $\varsigma_0^2 \in [0,\infty)$, we define:

$$\hat{h}_t(\theta) = \begin{cases} \varsigma_0^2 & t \leq 0 \\ \Phi_{t-1}(\hat{h}_{t-1}) & t \geq 1, \end{cases}$$

where $\Phi_t : \mathbb{C}(K,[0,\infty)) \to \mathbb{C}(K,[0,\infty))$ is defined by $[\Phi_t(s)](\theta) = g_\theta(X_t, s(\theta))$, $t \in \mathbb{Z}$. The $\hat{h}_t(\theta)$ is regarded as an estimate of σ_t^2 under the parameter hypothesis of θ. Note that $\hat{h}_t(\theta_0) = \hat{\sigma}_t^2$ for all $t \in \mathbb{N}$. The following is due to Proposition 3.12 of Straumann and Mikosch (2006).

Proposition 2. *Assume that the map $(\theta, s) \mapsto g_\theta(x, s)$ is continuous at every $x \in \mathbb{R}$. Then, if* $\mathrm{E}(\log^+|\Phi_0(\varsigma_0^2)|) < \infty$, $\mathrm{E}[\log^+ \Lambda(\Phi_0)] < \infty$, $\exists$ *integer $r \geq 1$ such that* $\mathrm{E}[\log \Lambda(\Phi_0^{(r)})] < 0$, *then the SRE $s_{t+1} = \Phi_t(s_t)$ has a unique ergodic and stationary solution (h_t). Moreover, h_t is $\mathcal{F}_{t-1}$-measurable and $h_t(\theta_0) = \sigma_t^2$ a.s.*

Suppose that $X_0, \ldots, X_n$ are generated by model (1) with true parameter θ_0. We define the quasi likelihood by

$$\hat{L}_n(\theta) = \sum_{t=1}^n \hat{l}_t(\theta) = -\frac{1}{2}\sum_{t=1}^n \left(\frac{X_t^2}{\hat{h}_t(\theta)} + \log \hat{h}_t(\theta)\right). \tag{3}$$

Then, the QMLE $\hat{\theta}_n$ is given by

$$\hat{\theta}_n = \arg\max_{\theta\in K} \hat{L}_n(\theta). \tag{4}$$

Similarly, we define

$$L_n(\theta) = \sum_{t=1}^n l_t(\theta) = -\frac{1}{2}\sum_{t=1}^n \left(\frac{X_t^2}{h_t(\theta)} + \log h_t(\theta)\right) \tag{5}$$

and

$$\tilde{\theta}_n = \arg\max_{\theta\in K} L_n(\theta). \tag{6}$$

Below is the list of the conditions for the strong consistency of $\hat{\theta}_n$:
(C1) Model (1) with $\theta = \theta_0$ admits a unique stationary ergodic solution (X_t, σ_t^2) with $\mathrm{E}(\log^+ \sigma_0^2) < \infty$.
(C2) The conditions in Proposition 2 are satisfied with $\theta_0 \in K$.
(C3) There exists a constant $\underline{g} > 0$ such that $g_\theta(x, s) \geq \underline{g}$ for all $(x, s) \in \mathbb{R} \times [0,\infty)$ and $\theta \in K$.
(C4) $\forall\theta \in K$, $h_0(\theta) \equiv \sigma_0^2$ a.s. if and only if $\theta = \theta_0$.

Moreover, we assume the following conditions to get the asymptotic normality of the QMLE:
(N1) h_t is twice continuously differentiable on K.
(N2) The following moment conditions hold: (i) $\mathrm{E}Z_0^4 < \infty$ (ii) $\mathrm{E}\left(\frac{\|h_0'(\theta_0)\|^2}{\sigma_0^4}\right) < \infty$, (iii) $\mathrm{E}\|l_0'\|_K < \infty$, (iv) $\mathrm{E}\|l_0''\|_K < \infty$.
Here, the derivatives are the same as those of Straumann and Mikosch (2006) and $\|\cdot\|$ stands for the Frobenius norm with $\|g\|_K = \sup_{s\in K}\|g(s)\|$.
(N3) The components of the vector $\frac{\partial g_\theta}{\partial\theta}(X_0, \sigma_0^2)|_{\theta=\theta_0}$ are linearly independent random variables.

The following is due to Theorem 7.1 of Straumann and Mikosch (2006).

Proposition 3. *Under conditions* **(C1)–(C4)** *and* **(N1)–(N3)**, *the QMLE $\hat{\theta}_n$ is strongly consistent and asymptotically normal, that is,*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(\mathbf{0}, \mathbf{V}_0), \quad as\ n \to \infty,$$

where the asymptotic covariance matrix $\mathbf{V}_0 = \frac{1}{4}\mathrm{E}(Z_0^4 - 1)(\mathrm{E}[(\frac{\partial\sigma^2(\theta_0)}{\partial\theta_0})^T (\frac{\partial\sigma^2(\theta_0)}{\partial\theta_0})/\sigma_0^4])$.

For example, the AGARCH(1,1) model, given by

$$X_t = \sigma_t Z_t, \tag{7}$$
$$\sigma_t^2 = \omega + \alpha(|X_{t-1}| - \gamma X_{t-1})^2 + \beta\sigma_{t-1}^2,$$

fulfills the conditions in the propositions. Let $\theta = (\omega, \alpha, \beta, \gamma)^T$ with $\omega > 0$, $\alpha, \beta \geq 0$ and $|\gamma| \leq 1$. The AGARCH(1,1) model has a unique ergodic and stationary solution if and only if

$$\lim_{n\to\infty} \frac{1}{n+1}\mathrm{E}\left[\sum_{t=-n+1}^{0} \log(\alpha(|Z_t| - \gamma Z_t)^2 + \beta)\right] < 0.$$

Further, $\alpha\mathrm{E}[(|Z_0| - \gamma Z_0)^2] + \beta < 1$ implies $\mathrm{E}X_0^2 < \infty$. If the distribution of Z_0 is not concentrated at two points and $\theta_0 \in K$, the QMLE is strongly consistent. Furthermore, if $\mathrm{E}Z_0^4 < \infty$ and $P(|Z_0| \leq z) = o(z^\mu)$ for some $\mu > 0$ as $z \downarrow 0$, then the QMLE is asymptotically normal with covariance matrix $V_0 = \frac{1}{4}\mathrm{E}(Z_0^4 - 1)\mathrm{E}[(\frac{\partial\sigma^2(\theta_0)}{\partial\theta_0})^T(\frac{\partial\sigma^2(\theta_0)}{\partial\theta_0})/\sigma_0^4]$.

3. CUSUM monitoring procedure

We apply the monitoring method proposed by Gombay and Serban (2009) to model (1). Below, we deal with the monitoring procedure to detect a change of θ_1. The other parameters can be handled similarly. For this task, we set up the null hypothesis:

$H_0 : \theta_{01}$ remains unchanged up to time n.

Here, θ_{01} is assumed to be known and $\eta = (\theta_2, \ldots, \theta_d)^T$ plays as an unknown nuisance parameter, assumed to be constant up to time n.

For each $k \geq 1$, we define

$$W_k(\theta_{01}, \hat{\eta}_n) = \hat{V}_{n,1,1}^{-1/2}(\theta_{01}, \hat{\eta}_n)\sum_{t=1}^k \frac{\partial\hat{l}_t(\theta_{01}, \hat{\eta}_n)}{\partial\theta_1}, \tag{8}$$

with $\hat{V}_{n,1,1}^{-1}(\theta)$ stands for the inverse of the (1,1)th entry of the matrix:

$$\hat{V}_n(\theta) = \left(\frac{1}{4n}\sum_{t=1}^n(\hat{Z}_t^4 - 1)\right)\left(\frac{1}{n}\sum_{t=1}^n \frac{\hat{h}_t'(\theta)^T\hat{h}_t'(\theta)}{\hat{h}_t(\theta)^2}\right)^{-1}, \tag{9}$$

with $\hat{Z}_t = X_t/(\hat{h}_t(\theta)^{1/2})$.