Economics Letters 123 (2014) 113–117

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Relaxing the substitutes condition in matching markets with contracts

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HIGHLIGHTS

- New necessary condition for existence of a stable allocation (matching with contracts).
- New sufficient conditions for existence of a stable allocation.
- Conditioning on feasible worker preferences allows for new firm preferences.

ARTICLE INFO

Article history:
Received 18 October 2013
Received in revised form
17 January 2014
Accepted 31 January 2014
Available online 6 February 2014

JEL classification:
C62
C78
D47

Keywords:
Matching
Stability
Consistent substitutes

ABSTRACT

In the many-to-one matching model with contracts, I provide new necessary and new sufficient conditions for the existence of a stable allocation. These new conditions exploit the fact that one side of the market has strict preferences over individual contracts.

1. Introduction

Throughout the literature on many-to-one matching, the assumption is often made that preferences satisfy a gross substitutes condition. This condition requires that if we are in a market comprised of firms and workers, firms must treat individual workers (or contracts) as substitutes: a firm would never reject an offer from a worker when it is part of a certain set, only to accept the same offer when it is part of a larger set. The contribution of the present paper is to introduce new restrictions on preferences which are weaker than substitutability, but which are either necessary or sufficient for the existence of a stable allocation in the many-to-one market with contracts.[1]

The literature already contains several sufficient conditions which are strictly weaker than substitutability, for example bilateral substitutability (BLS) (Hatfield and Kojima, 2010), and at least one necessary condition, weak substitutability (WS) (Hatfield and Kojima, 2008). I respectively relax and strengthen these known conditions by using the fact that each worker has strict preferences over individual contracts, which is information that is often left unexploited in the past literature. I introduce a new necessary condition, consistent substitutability (CS), which eliminates the possibility of instabilities which may arise when preferences do not satisfy BLS. CS is a stronger necessary condition than WS; however it is not clear whether it is sufficient. I also introduce several new sufficient conditions for the existence of a stable allocation, the weakest of which is cumulative offer revealed bilateral substitutability (CBLS). While BLS is a condition which ensures that the cumulative offer process introduced by Hatfield and Milgrom (2005) always results in a stable allocation, CBLS weakens BLS by allowing a firm's preferences to fail BLS, as long as the cumulative

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[1] This model was introduced, in various forms, by Kelso and Crawford (1982), Roth (1984), and Hatfield and Milgrom (2005).

http://dx.doi.org/10.1016/j.econlet.2014.01.040

offer process results in a stable allocation for any feasible collection of worker preference profiles.

2. Model

There is a finite set of workers W, a finite set of firms F, and a finite set of contracts X. Each contract $x \in X$ is a triplet (f, w, θ), where $f \in F$ is a firm, $w \in W$ is a worker, and $\theta \in \Theta$ is an extra parameter, or set of parameters, over which both the worker and the firm named in the contract may have preferences over. For example, a common interpretation of θ would be a wage or salary, but it could also be work hours, insurance plans, etc.[2] I may refer to agent a named in contract x as $a(x)$. Let $W(X') = \cup_{x \in X'}\{w(x)\}$ be the collection of workers named in some contract in set X'.

Firms can sign multiple contracts, but each worker can sign at most one contract. A set of contracts $X' \subseteq X$ is *feasible* if each worker in W is named in at most one contract in X'. Each worker w has preferences, $\succeq_w$, which are complete, transitive, and strict over each contract $x \in X$ which names her, and each firm f has preferences, $\succeq_f$, which are strict over feasible sets of contracts $X' \subseteq X$ for which $f = f(x) \forall x \in X'$. Each agent has a choice function, $Ch_a(\cdot)$, which agrees with the agent's preferences. The *rejected set*, $R_a(X') = X' \setminus Ch_a(X')$, is the complement of $Ch_a(X')$ in X'. Each side of the market has a collective choice function defined by $Ch_F(X') = \cup_{f \in F} Ch_f(X')$ and $Ch_W(X') = \cup_{w \in W} Ch_w(X')$, respectively.

A set of contracts X' is an *allocation* if it is feasible. Agents who do not sign any contracts in X' are *unmatched*, and a contract or set of contracts X is *acceptable* if $Ch_a(X) = X$. An allocation X' is *stable* if it is both:

1. *Individually Rational*: $Ch_W(X') = Ch_F(X') = X'$.
2. *Unblocked*: There does not exist a firm f and set of contracts $X'' \neq Ch_f(X')$ such that $X'' = Ch_f(X' \cup X'') \subseteq Ch_W(X' \cup X'')$.

3. Results on preferences

As noted, one common restriction on preferences which is sufficient for the existence of a stable allocation is that all contracts are substitutes for all firms. Formally, contracts are *substitutes* for f if for all subsets of X such that $X' \subset X'' \subseteq X$ we have $R_f(X') \subseteq R_f(X'')$. This condition has many nice properties, notably that it guarantees that the set of stable allocations is a lattice when using either side of the market's collective preferences as a partial order. However this condition is not necessary for the existence of a stable allocation.[3]

Hatfield and Kojima (2008) show that one condition which is necessary for the existence of a stable allocation is weak substitutability. Contracts are *weak substitutes* for f if $R_f(X') \subseteq R_f(X'')$ for all $X' \subset X'' \subseteq X$ such that if $x, x' \in X''$ and $w(x) = w(x')$, then $x = x'$. This condition requires that contracts are substitutes when firms are offered sets of contracts in which no worker is named in multiple contracts. The following example shows that WS is not sufficient to guarantee a stable allocation exists.

Example 1. Let there be two firms, f and g, and three workers w_1, w_2, and w_3, with preferences[4]:

$$\succeq_f : \{x, y, z\} \succ_f \{y'\} \succ_f \{z'\} \succ_f \{x, y\} \succ_f \{x, z\} \succ_f \{y, z\} \succ_f \{x\} \succ_f \{y\} \succ_f \{z\}$$

$$\succeq_g : \{x'\} \succ_g \{z''\}$$

$$\succeq_{w_1} : x \succ_{w_1} x'$$

$$\succeq_{w_2} : y \succ_{w_2} y'$$

$$\succeq_{w_3} : z' \succ_{w_3} z'' \succ_{w_3} z.$$

Contracts are weak substitutes for both firms, but not substitutes for f. No stable allocation exists.

To see the failure of stability in Example 1, imagine the following sequence of events: workers w_1 and w_2 initially sign contracts x and y, but then w_3 blocks this allocation by offering f contract z'. Having nowhere else to go, w_2 counters by offering f contract y', while w_1 leaves f to sign contract x' with g. Now w_3 is unmatched and unable to unilaterally make an acceptable offer to either firm, and therefore she is willing to sign contract z when it is offered along with contracts x and y to f. However, once that set of contracts is signed, w_3 can now successfully offer z'' to g, and the market is back to its initial allocation.

Notice specifically the sequence of sets of contracts that f signs in this scenario:

$$\{X_1, X_2, X_3, X_4\} = \{\{x, y\}, \{z'\}, \{y'\}, \{x, y, z\}\}.$$

This sequence is such that for each element X_n, $Ch_f(X_1 \cup \cdots \cup X_n) = X_n$, and $X_n \subseteq Ch_W(X_{n-1} \cup X_n)$. I will define a "blocking sequence" as a sequence which exhibits these properties, but first I must introduce some notation. For the set of contracts X, let $\mathcal{P}_a(X)$ be the set of all feasible preference profiles for agent a. Let $\mathcal{P}_W(X) = \prod_{w \in W} \mathcal{P}_w(X)$, and $\mathcal{P}_F(X) = \prod_{f \in F} \mathcal{P}_f(X)$. For any element of $\mathcal{P}_W(X)_W \times \mathcal{P}_f(X)$, a *blocking sequence* $\{X_k\}_{k=1}^n$ is a sequence of sets of contracts such that:

B1. $X_{m+1} = Ch_f(\cup_{l=1}^{m+1} X_l)$ for all $m = 1, \ldots, n-1$, and
B2. $X_{m+1} \subseteq Ch_W(X_m \cup X_{m+1})$ for all $m = 1, \ldots, n-1$.

Using this definition, I introduce a new necessary condition for the existence of a stable allocation, consistent substitutability (CS). Consistent substitutability ensures that certain blocking sequences, such as in Example 1, cannot exist. Formally, contracts are *consistent substitutes* for f if, when there exist sets of contracts $X' \subset X''$ and $\{y\}$ such that:

- $W(X' \cup \{y\}) \cap W(Ch_f(X'')) = \emptyset$, and
- $X' \subset Ch_f(X'' \cup \{y\})$

then there does not exist a sequence $\{X_k\}_{k=1}^n$ which satisfies B1 and:

CS1. $X_1 = Ch_f(X'' \cup \{y\}) \setminus \{y\}$
CS2. $X_{n-1} = Ch_f(X'')$
CS3. $X_n = Ch_f(X'' \cup \{y\})$
CS4. For every w named in a contract in the sequence, there exists a total order $\geq_w$ over sets containing a contract naming w which satisfies (i) $X_l =_w X_{l+1}$ if w is named in the same contract in X_l and X_{l+1}, and (ii) $X_{l+1} >_w X_l$ if w is named in two different contracts in X_l and X_{l+1}.

Theorem 1. *If there are at least two firms in a market, f and g, and if contracts are not consistent substitutes for f, then there exist preference profiles for the workers in W, and a preference profile over individual contracts for g, such that no stable allocation exists.*

Intuitively what CS is doing is making sure that if there exists a sequence that satisfies B1, then it cannot satisfy B2, since any total order $\geq_w$ over the set of contracts which name w is one possible realization of w's preferences. The following examples help illustrate the meaning of CS and the theorem. Example 2 also shows that CS implies WS.

Example 2. Suppose contracts are not weak substitutes for f. Then, by the definition of WS, there exists a contract x' and sets of contracts Y and Y', with $Y \subset Y'$, such that $x' \in R_f(Y)$ and $x' \in$

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2 Echenique (2012) shows that a model in which θ is a salary and firms and workers have quasi-linear preferences is equivalent to a model in which all contracts are substitutes for all firms. Therefore, for motivating the results of the present paper, it may be better to think of each θ as something other than a salary.

3 Cf. Hatfield and Kojima (2008), Hatfield and Kojima (2010).

4 Throughout all examples in this paper, contracts x, x' name worker w_1, contracts y, y' name w_2, and z, z', z'' name w_3. The superscript denotes different contracts.