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Gross substitutes and complements: A simple generalization

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HIGHLIGHTS

- We generalize the gross substitutes and complements framework (Sun and Yang, 2006).
- We show that competitive equilibrium with indivisible goods exists under weaker conditions.
- We show how the competitive equilibrium allocation can be implemented.

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ABSTRACT

We extend the gross substitutes and complements framework (Sun and Yang, 2006). Competitive equilibrium with indivisible goods exists under significantly weaker, intuitive and interpretable conditions. A generalized dynamic double-track procedure (Sun and Yang, 2008, 2009) finds the competitive equilibrium outcome.

1. Introduction

In this paper, we study an exchange economy in which agents have heterogeneous preferences over indivisible items. The relationship between such exchange economies, auctions, and matching markets is already well known for the case of substitutable goods (Kelso and Crawford, 1982; Milgrom, 2000; Milgrom and Strulovici, 2009). However, in labor markets, firms may view skills of different workers are complementary. For example, a hospital may want to hire a surgeon together with an anesthetist. Technological complementarities also occur in many designed markets, such as telecommunications auctions. For instance, in the Japanese 4G spectrum auction, there were 10 lots of 20 MHz spectrum bands. There were two competing technologies: TDD and FDD. The FDD technology required paired lots – uplink and a downlink – which had to be located sufficiently far away from each other on the spectrum. For a firm wanting to deploy FDD, any (potential) uplink or any downlink bandwidth is substitutable (hence bundling is not trivial), but an uplink and a downlink band are complementary. On the other hand, TDD only required any one of (or several adjacent) substitutable spectrum band lots (Matsushima, 2012). However, it is well known that, in markets with indivisible commodities, competitive equilibrium does not always exist when complementarities are present (Kelso and Crawford, 1982; Gul and Stacchetti, 1999). This paper offers a new sufficient condition (satisfied for the Japanese spectrum auction) for the existence of competitive equilibrium in an exchange economy in which agents trade indivisible substitutable and complementary goods.

This papers builds on the *gross substitutes and complements* (GSC) preference framework introduced by Sun and Yang (2006). They showed that if, for example, a seller offers trousers and shirts and all buyers regard any two shirts (or any two pairs of trousers) as substitutes, but any shirt and pair of trousers as complements, then competitive equilibrium will exist in this economy when agents' utility functions are quasilinear in prices. In the present model, all goods can be partitioned into sets of substitutes and every buyer regards goods from some two partition elements as

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complements. As an example, consider an economy in which the seller offers three types of goods: jackets, trousers, and shirts. Buyers view any type of good as substitutes — this is a natural assumption when the goods of a particular type are sufficiently similar. There are also two types of buyers: a student who views jackets and trousers as complements, and a professor who views trousers and shirts as complements. We show that in this sort of economy competitive equilibrium is guaranteed to exist. However, if we add another agent into the economy – a post-doc who regards jackets and shirts as complements – then competitive equilibrium is no longer guaranteed to exist (see Example 2). This failure of equilibrium existence occurs because there is an *odd cycle* in the *generalized gross substitutes and complements* (GGSC) structure of agents' preferences: jackets and trousers are complements for the student, trousers and shirts are complements for the professor, and shirts and jackets are complements for the post-doc. In Section 3, we show that competitive equilibrium exists whenever these odd cycles are absent: a much weaker, yet intuitive, condition than those found previously.[1] Other generalizations of the GSC framework were proposed by Baldwin and Klemperer (2013) and Shioura and Yang (2013), but only the latter developed an auction procedure. In Teytelboym (2012), we show how to apply these results to multi-unit environments and trading networks.

2. Model

2.1. Ingredients

There is a finite set of agents $i \in I$ and a finite set of indivisible goods $\omega \in \Omega$ in the economy. Goods are partitioned into M (possibly empty) disjoint subsets (of similar goods) of Ω, forming a set $\mathcal{S} = \{S_1, \dots, S_M\}$ such that $S_n \cap S_m = \emptyset$ (where $n, m = 1, \dots, M$; $n \neq m$) and $\bigcup_{m=1}^{m=M} S_m = \Omega$. Each element of the partition represents a set of similar goods (such as shirts of a different color). Let $\Psi \in 2^{\Omega}$ be a bundle of goods and Ψ_i be a bundle for agent $i \in I$. Denote p_ω as the price of good ω and $p \in \mathbb{R}^{|\Omega|}$ as the price vector. An allocation is a partition Π of goods into (possibly empty) bundles for different agents ($\Pi = \{\Psi_i\}_{i \in I}$ such that $\bigcup_{i \in I} \Psi_i = \Omega$ and $\Psi_i \cap \Psi_j = \emptyset$). An arrangement is a pair $[\Pi, p]$, which associates prices to all goods in the economy. We assume that agents' net utility functions are quasilinear in prices

$$U_i([\Pi; p]) \equiv u_i(\Psi_i) - \sum_{\omega \in \Psi_i} p_\omega \tag{1}$$

where $u_i : 2^{\Omega} \to \mathbb{R}_+$ is a weakly increasing valuation function with $u_i(\emptyset) = 0$ and agents are not subject to any liquidity or budget constraints.[2] Therefore, the trading economy can be described by the set of goods and the agents' valuations of every bundle: $\mathcal{E} \equiv \{\Omega, (u_i, i \in I)\}$. The demand correspondence $D_i : \mathbb{R}^{|\Omega|} \to 2^{\Omega}$ for agent i is defined in the usual way

$$D_i(p) \equiv \arg\max_{\Psi \subseteq \Omega} U_i([\Psi; p]). \tag{2}$$

A competitive equilibrium in this economy consists of an allocation of goods to agents and a vector of competitive prices, such that the market clears: every agent demands precisely his allocation at this price vector.

Definition 1. *Competitive equilibrium* is an arrangement $[\Pi; p]$ such that for all $i \in I$, $\Psi_i \in D_i(p)$.

[1] The "no odd party" (Tan, 1991) and "no-odd-rings" conditions (Chung, 2000) guarantee existence of stable matchings in the roommate market. Gudmundsson (2013) showed that absence of odd cycles in a certain linear programming problem guarantees existence of equilibrium in the partnership formation problem (Talman and Yang, 2011). However, these results for one-sided matching problems are logically unrelated to the present chapter.

[2] These assumptions are necessary for the results and relaxing them is an interesting open problem.

2.2. Preferences

First, we formally define preferences that satisfy gross substitutes and complements (Sun and Yang, 2006). Let us consider $\mathcal{S} = \mathcal{S}^* = \{S_1, S_2\}$, which represents shirts (S_1) and trousers (S_2). For notation purposes, define $e(k)$ is a kth unit vector in $\mathbb{R}^{|\Omega|}$ and $A^c = \Omega \setminus A$.

Definition 2 (*Sun and Yang, 2006*)**.** Preferences of agent i satisfy *gross substitutes and complements (GSC)* on $\mathcal{S}^*$ if for any $p \in \mathbb{R}^{|\Omega|}$, $\omega_k \in S_m$, $\delta \geq 0$, $A \in D_i(p)$, there exists $B \in D_i(p + \delta e(k))$ such that $[A \cap S_m] \backslash \{\omega_k\} \subseteq B$ and $A^c \cap S_m^c \subseteq B^c$.

In other words, preferences of agent i satisfy GSC if whenever the price of a shirt (trousers) increases, i's resulting demand, B, for other shirts (trousers) does not fall i.e $[A \cap S_m] \backslash \{\omega_k\} \subseteq B$, and demand for trousers (shirts) does not rise i.e. $A^c \cap S_m^c \subseteq B^c$. Using our example in the Introduction, the professor has GSC preferences over shirts and trousers. Sun and Yang (2006) show that whenever preferences of all agents satisfy GSC competitive equilibrium exists. We now turn to the main assumption on individual preferences and structure of the economy which generalizes GSC.

Definition 3. Preferences satisfy *generalized gross substitutes and complements (GGSC)* on $\mathcal{S}$ if for any $p \in \mathbb{R}^{|\Omega|}$, $\omega_k \in S_m$, $\delta \geq 0$, $A \in D_i(p)$, $S_m \in \mathcal{S}$ and $i \in I$, there exists one $S_n \in \mathcal{S}$ and $B \in D_i(p + \delta e(k))$ such that $[A \cap S_m] \backslash \{\omega_k\} \subseteq B$, $A^c \cap S_n \subseteq B^c$ and $A \cap [S_m \cup S_n]^c = B \cap [S_m \cup S_n]^c$.

In words, agents' demand correspondences have GGSC structure if we can divide goods into a partition $\mathcal{S}$ (for all agents) such that, whenever we consider preferences over goods contained in any two elements of $\mathcal{S}$ *in isolation*, these preferences satisfy GSC for some agents. From now on, whenever we say that agents have GSC preferences over S_m and S_n, we will mean that these agents would have GSC preferences over $\mathcal{S}^* = \{S_m, S_n\}$ if the goods in S_m and S_n were considered *in isolation*. Different agents may have GSC preferences over different pairs of the partition elements of $\mathcal{S}$.[3] Again returning to our example from the Introduction, the student has GSC preferences over jackets and trousers and the post-doc has GSC preferences over jackets and shirts: so the agents' preferences satisfy GGSC. It is worth noting that, for agents who have GSC preferences over S_m and S_n, changes in prices for a good contained in S_m do not have any effect on the demands for goods outside $S_n \cup S_m$, and changes in prices for goods contained $[S_n \cup S_m]^c$ do not affect the demands for goods in $S_n \cup S_m$. In other words, for these agents, goods in $S_n \cup S_m$ and $[S_n \cup S_m]^c$ are independent. To see how the GGSC structure generalizes GSC, note that preferences with a GGSC structure satisfy GSC for all agents if $M = 2$. Since we do not rule out that $S = \emptyset$, it is clear that if $M = 2$ and $\mathcal{S} = \{S, \emptyset\}$, we return to the *gross substitutes* framework of Kelso and Crawford (1982).[4]

2.3. Motivating examples

Will a competitive equilibrium exist in a trading economy where agents' preferences satisfy GGSC?

Example 1. Consider a trading economy with four buyers i, j, k, l, a seller s, and four goods $\{\omega_1, \omega_2, \omega_3, \omega_4\}$. The seller's values are zero

[3] We could allow the same agent to have GSC preferences over several pairs of elements of $\mathcal{S}$ (because of the quasilinearity of the utility functions), but, while this complicates the exposition, it does not affect the results in any way.

[4] Therefore, we do not rule out that some agents may only view goods within an element $\mathcal{S}$ as substitutes and no goods as complements.