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The nucleolus of large majority games[☆]

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HIGHLIGHTS

- A bound for the distance between weights and the nucleolus.
- Coincidence of the nucleolus and weights for non-homogeneous games.
- A limit theorem for the nucleolus, similar to Penrose's limit theorem.
- A sufficient criterion for a positive nucleolus for non-null players.

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ABSTRACT

Members of a shareholder meeting or legislative committee have greater or smaller voting power than meets the eye if the nucleolus of the induced majority game differs from the voting weight distribution. We establish a new sufficient condition for the weight and power distributions to be equal, and we characterize the limit behavior of the nucleolus in case all relative weights become small.

1. Introduction

Among all individually rational and efficient payoff vectors in a game v with transferable utility, the *nucleolus* selects a particularly stable one. It quantifies each coalition's dissatisfaction with a proposed vector x as the gap between the coalition's worth $v(S)$ and the surplus share $\sum_{i\in S} x_i$ that is allocated to members of $S \subseteq N$; then it selects the allocation x^* which involves lexicographically minimal dissatisfaction. In contrast to other prominent point solutions in cooperative game theory, such as the Shapley value, x^* is guaranteed to lie in the *core* of game (N, v) whenever that is non-empty.

Even before the final version of Schmeidler's article which established the definition, existence, uniqueness, and continuity of the nucleolus was published in 1969, Peleg (1968) had applied it to *weighted majority games (WMG)*. In these games the worth of a coalition S of players is either 1 or 0, i.e., S is either winning or losing, and there exists a non-negative quota-and-weight *representation* $[q; w_1, \ldots, w_n]$ such that $v(S) = 1$ iff $\sum_{i\in S} w_i \geq q$. The weight vectors that constitute a representation of a given WMG v for some quota q form a non-singleton convex set $R(v)$.

Peleg highlighted a property of constant-sum WMGs with a *homogeneous* representation, i.e., one where total weight of any minimal winning coalition equals q: the nucleolus x^* of such a WMG v is contained in $R(v)$, i.e., it is also a representation.[1] Despite this early start, the relation between voting weights and the nucleolus of

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[1] A WMG (N, v) is called *constant-sum* if for any $S \subseteq N$ either $v(S) = 1$ or $v(N \setminus S) = 1$. $S \subseteq N$ is a *minimal winning coalition (MWC)* if $v(S) = 1$ and $v(T) = 0$ for any $T \subset S$.

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weighted majority games – constant-sum or not, homogeneous or inhomogeneous – has to the best of our knowledge not been studied systematically so far. This paper is a first attempt to fill this gap.

Discrepancies between weights and the nucleolus matter because the nucleolus is an important indicator of influence in collective decision bodies. It emerges as an equilibrium price vector in models that evaluate voters' attractiveness to competing lobbying groups (see Young, 1978; Shubik and Young, 1978); more recent theoretical work by Montero (2013, 2006) has established it as a focal equilibrium prediction for strategic bargaining games with a majority rule.[2] So large differences between a voter i's weight w_i and nucleolus x_i^* can mean that the real power distribution in a decision body such as a shareholder meeting is hidden from the casual observer. This intransparency can be particularly problematic for political decision bodies, where voting weight arrangements affect the institution's legitimacy.[3]

This paper investigates absolute and relative differences between players' relative voting weights as defined by vote shares in an assembly, electoral college, etc., and the nucleolus of the implied WMG. We determine an upper bound on their $\|\cdot\|_1$-distance which depends only on quota and maximum weight in a given representation in Lemma 1. The lemma allows us to conclude that if the relative weight of every individual voter in a player set $\{1,\ldots,n\}$ tends to zero, then the ratio x_i^*/x_j^* of two nucleolus components converges to w_i/w_j for all regular voters i and j as $n\to\infty$ (Proposition 1). This complements analogous limit results in the literature on the Shapley value, the Banzhaf value and voter pivotality on intervals (see Neyman, 1982; Lindner and Machover, 2004; Kurz et al., 2013) as well as for stationary equilibrium payoffs in legislative bargaining games à la Baron–Ferejohn (see Snyder et al., 2005). We also establish a new sufficient condition for the nucleolus to coincide with given relative weights (Proposition 2). It implies that a finite number of replications brings about full coincidence for any given WMG.

2. Nucleolus

Consider a WMG (N, v) with representation $[q; w_1,\ldots,w_n]$. Using notation $x(S)=\sum_{i\in S}x_i$, a vector $x\in\mathbb{R}^n$ with $x_i\geq v(\{i\})$ and $x(N)=v(N)$ is called an *imputation*. For any coalition $S\subseteq N$ and imputation x, call $e(S,x)=v(S)-x(S)$ the *excess* of S at x. It can be interpreted as quantifying the coalition's dissatisfaction and potential opposition to an agreement on allocation x. For any fixed x let $S_1,\ldots,S_{2^n}$ be an ordering of all coalitions such that the excesses at x are weakly decreasing and denote these ordered excesses by $E(x)=\left(e(S_k,x)\right)_{k=1,\ldots,2^n}$. Imputation x is *lexicographically less* than imputation y if $E_k(x)<E_k(y)$ for the smallest component k with $E_k(x)\neq E_k(y)$. The *nucleolus* of (N,v) is then uniquely defined as the lexicographically minimal imputation.[4]

As an example, consider (N,v) with representation $[q;w]=[8;6,4,3,2]$. The nucleolus can be computed as $x^*=(2/5,1/5,1/5,1/5)$ by solving a sequence of linear programs—or by appealing to the sufficient condition of Peleg (1968) after noting that the game is constant-sum and permits a homogeneous representation $[q';w']=[3;2,1,1,1]$. Denoting the *normalization* of weight vector w by $\bar{w}$, i.e., $\bar{w}=w/\sum w_i$, the respective total differences between relative weights and the nucleolus are $\|\bar{w}-x^*\|_1=2/15$ for the first and $\|\bar{w}'-x^*\|_1=0$ for the second representation (with $\|x\|_1=\sum|x_i|$).

3. Results

Saying that representation $[q;w]$ is *normalized* if $w=\bar{w}$, we have[5] the following lemma.

Lemma 1. *Consider a normalized representation $[q;w]$ with $0<q<1$ and $w_1\geq\cdots\geq w_n\geq 0$ and let x^* be the nucleolus of this WMG. Then*

$$\|x^*-w\|_1\leq\frac{2w_1}{\min\{q,1-q\}}. \tag{1}$$

If we consider a sequence $\{(\{1,\ldots,n\},v^{(n)})\}_{n\in\mathbb{N}}$ of n-player WMGs with representations $[q^{(n)};w^{(n)}]$ such that the normalized quota $\bar{q}^{(n)}$ is bounded away from 0 and 1 (or, more generally, 0 and 1 are no cluster points of $\{\bar{q}^{(n)}\}_{n\in\mathbb{N}}$), and each player i's normalized weight $\bar{w}_i^{(n)}$ vanishes as $n\to\infty$ then Lemma 1 implies

$$\lim_{n\to\infty}\|x^{*(n)}-\bar{w}^{(n)}\|_1\to 0. \tag{2}$$

Convergence to zero of the total difference between nucleolus components $x_i^{*(n)}$ and relative voting weights $\bar{w}_i^{(n)}$ does not yet guarantee that the nucleolus is asymptotically proportional to the weight vector, i.e., that each ratio $x_i^{*(n)}/x_j^{*(n)}$ converges to w_i/w_j. This can be seen, e.g., by considering

$$\left[q^{(n)};w^{(n)}\right]=\left[\frac{2n-1}{2};1,\underbrace{2,\ldots,2}_{n-1}\right]. \tag{3}$$

The nucleolus either equals $\left(0,\frac{1}{n-1},\ldots,\frac{1}{n-1}\right)$ or $\left(\frac{1}{n},\ldots,\frac{1}{n}\right)$ depending on whether n is even or odd; ratio $x_1^{*(n)}/x_2^{*(n)}\neq\frac{1}{2}$ alternates between 0 and 1.

But such pathologies are ruled out for players i and j whose weights are "non-singular" in the weight sequence $\left\{w^{(n)}\right\}_{n\in\mathbb{N}}$. Specifically, denote the total number of players $i\in\{1,\ldots,n\}$ with an identical weight of $w_i^{(n)}=\omega$ by $m_\omega(n)$, and their relative number by $\bar{m}_\omega(n)=\omega_n(n)/n$. We say that player j with weight w_j is *regular* if $\bar{m}_{w_j}(n)\cdot\bar{w}_j^{(n)}$ is bounded away from 0 by some constant $\varepsilon>0$. Lemma 1 then implies[6] the following proposition.

Proposition 1. *Consider a sequence $\left\{\left[q^{(n)};(w_1,\ldots,w_n)\right]\right\}_{n\in\mathbb{N}}$ with corresponding normalized quotas that exclude 0 and 1 as cluster points and with normalized weights satisfying $\bar{w}_k^{(n)}\downarrow 0$ for every $k\in\mathbb{N}$ as $n\to\infty$. Then the nucleolus $x^{*(n)}$ of the WMG represented by $\left[q^{(n)};(w_1,\ldots,w_n)\right]$ satisfies*

$$\lim_{n\to\infty}\frac{x_i^{*(n)}}{x_j^{*(n)}}=\frac{w_i}{w_j} \tag{4}$$

for any regular players i and j.

2 Corresponding experimental lab evidence is mixed; see Montero et al. (2008). Non-cooperative foundations of the nucleolus for other than majority games have been given, e.g., by Potters and Tijs (1992) and Serrano (1993, 1995).

3 See Le Breton et al. (2012) for nucleolus-based power analysis of the European Union's Council; an early-day weight arrangement meant that Luxembourg had a relative voting weight of 1/17 but zero voting power.—In general, the power-to-weight ratio can differ arbitrarily from 1. For instance, the nucleolus of the WMG with representation $[0.5;(1-\varepsilon)/2,(1-\varepsilon)/2,\varepsilon]$ is $x^*=(1/3,1/3,1/3)$ for any $\varepsilon\in(0;0.5)$.

4 Schmeidler's (1969) original definition did not restrict the considered vectors to be imputations but is usually specialized this way. The set of imputations that minimize just the largest excess, $E_1(x)$, is called the *nucleus* of (N,v) by Montero (2006). Our results are stated for the nucleolus but apply to every element of the nucleus: both coincide under the conditions of Proposition 2; Lemma 1 and Proposition 1 generalize straightforwardly.

5 All proofs are provided in Mathematical appendix.

6 We assume $w_j^{(n)}=w_j$ in our exposition. Adaptations to cases where $q^{(n)}$ and $w_j^{(n)}$ vary in n are straightforward. The essential regularity requirement is that a voter type's aggregate relative weight does not vanish.