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Cumulative Prospect Theory for piecewise continuous distributions

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ABSTRACT

We extend the continuous Cumulative Prospect Theory by considering piecewise continuous distributions with a finite number of jump discontinuities. Such distributions are always relevant when outcomes depend on continuously distributed random variables and the dependency is defined by a piecewise continuous function. For example, such outcomes occur within the framework of financial engineering. We show how to apply the model to a broad class of piecewise continuous outcome functions that includes outcomes of guarantee certificates.

1. Introduction

Most literature dealing with a subjective evaluation of outcomes on the basis of the Prospect Theory (PT), as suggested by Kahneman and Tversky (1979), and Cumulative Prospect Theory (CPT), as proposed by Tversky and Kahneman (1992), focuses on the discrete modeling of probability distributions. However, in many cases, a more precise description of real decision making requires the generalization of evaluation approaches to continuous distributions. Therefore, Rieger and Wang (2008) extended PT to continuous distributions. Additionally, the extension of CPT to continuous distributions does not cause any theoretical problems from a theoretical point of view (see e.g., Davies and Satchell (2007)).[1] However, the distribution of outcomes is often only piecewise continuous, with a finite number of jump discontinuities. For example, such outcome distributions occur in financial engineering, a relevant field in behavioral economics,[2] which deals with customer-oriented design of financial securities.[3]

Even if implementation of piecewise continuous distributions is unproblematic on the basis of the continuous PT of Rieger and Wang (2008), implementation on the basis of continuous CPT is analytically and numerically difficult to handle

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[1] However, it must be pointed out that the use of continuous distributions in CPT evaluations can also be associated with calculation problems. See, e.g., Zou and Zagst (2017), chapter 5, who analyze optimal investment problems under consideration of transaction costs within the framework of CPT using (discrete) binomial return distributions as well as continuous return distributions.

[2] See, e.g., Breuer and Perst (2007) for a justification of the design of financial securities in the sense of behavioral financial engineering.

[3] See, e.g., Breuer and Perst (2007), p. 828.

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because the necessary derivatives are unavailable. Against this background, we extend the CPT by considering piecewise continuous distributions. The resulting model is easy to implement and enables the development of a CPT evaluation formula for piecewise continuous distributions of a broad class of outcomes.

2. Cumulative Prospect Theory for piecewise continuous distributions

In contrast to traditional expected utility theory (EUT), CPT decision makers evaluate possible outcomes $\tilde{Z}$ in relation to a reference point z^{ref}, i.e., CPT relates to relative outcomes defined as $\tilde{X} := \tilde{Z} - z^{ref}$. In the following, we assume $\tilde{Z}$ and therefore $\tilde{X}$ to be continuous outcome random variables. Let $f_{\tilde{X}}$ be the probability density function and $F_{\tilde{X}}$ be the probability distribution function of $\tilde{X}$. Further, relative outcomes are evaluated on the basis of a value function v. Because of loss aversion, individuals pay more attention to losses (negative relative outcomes) than gains (positive relative outcomes) implying that v is steeper for losses. In addition, we assume v to be continuously differentiable at x for all $x \neq 0$ and $\lim_{x \to 0-} v'(x)$ and $\lim_{x \to 0+} v'(x)$ to exist and to be positive and finite.[4] According to Davies and Satchel (2007), CPT evaluation for a continuous lottery $X^c := (\tilde{X}, f_{\tilde{X}})$ is expressed as

$$CPT(X^c) = \int_{-\infty}^{0} v(x) \cdot \frac{d}{dx} w^{-}\left(F_{\tilde{X}}(x)\right) dx - \int_{0}^{\infty} v(x) \cdot \frac{d}{dx} w^{+}\left(1 - F_{\tilde{X}}(x)\right) dx \tag{1}$$

where $w^+ : [0,1] \to [0,1]$ and $w^- : [0,1] \to [0,1]$ describe strictly increasing weighting functions with $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$.

Obviously, numerical problems when implementing the CPT are not considerable if distribution $F_{\tilde{X}}$ is continuous and $F_{\tilde{X}}(x) \in (0, 1)$ for all x. A problem arises if distribution $F_{\tilde{X}}$ is piecewise continuous with a finite number of jump discontinuities or if $F_{\tilde{X}}(x) = 0$ or $F_{\tilde{X}}(x) = 1$ for at least an $x \in (0, 1)$. The reason is that, in the case of jump discontinuities, the first derivatives of the function compositions $w^- \circ F_{\tilde{X}}$ and $w^+ \circ (1 - F_{\tilde{X}})$ do not exist in the whole domain.

In the following,[5] we show how to handle these problems on the basis of a general class of probability distribution functions $F_{\tilde{X}}$, which are defined for all $x \in \Re$ according to

$$F_{\tilde{X}}(x) = \sum_{i=0}^{n+1} \alpha_i \cdot H(x - \xi_i) + \sum_{j=1}^{m} \beta_j \cdot F_{\tilde{S}}\left(\phi_j(x)\right) \cdot H\left(x - \psi_j\right) \tag{2}$$

where $n, \ m \in IN$, $\alpha_i, \beta_j \in \Re$, $\xi_i, \psi_j \in \Re \cup \{-\infty\}$, $\varphi_j : \Re \to \Re$ are differentiable functions ($i = 0, \ldots, n+1$; $j = 1, \ldots, m$); $F_{\tilde{S}} : \Re \to (0, 1)$ is a differentiable probability distribution of a continuous random variable $\tilde{S}$ and H is the Heaviside function.[6] Furthermore, we assume $\xi_0 < \xi_i < \xi_{n+1}$ for all $i = 1, \ldots, n$, $\xi_0 \le \psi_j \le \xi_{n+1}$ for all $j = 1, \ldots, m$; and $F_{\tilde{X}}(x) = 1$ for $x \ge \xi_{n+1}$. This immediately implies $F_{\tilde{X}}(x) = 0$ for $x < \xi_0$. If $\alpha_0 = 0$ and $\alpha_{n+1} = 0$ we set $\xi_0 = -\infty$ and $\xi_{n+1} = +\infty$, respectively. Altogether, we consider a broad class of distributions, including piecewise continuous functions with a finite number of jump discontinuities.

Because $F_{\tilde{X}}(x) = 0$ for all $x \in (-\infty, \xi_0)$, $F_{\tilde{X}}(x) \in (0, 1)$ for all $x \in [\xi_0, \xi_{n+1})$, and $F_{\tilde{X}}(x) = 1$ for all $x \in [\xi_{n+1}, \infty)$ (with $(-\infty, -\infty) := [\infty, \infty) := \emptyset$ and $[-\infty, \xi_{n+1}) := (-\infty, \xi_{n+1})$), we can transform $F_{\tilde{X}}$ according to

$$F_{\tilde{X}}(x) = H(x - \xi_{n+1}) + \hat{F}_{\tilde{X}}(x) \cdot (H(x - \xi_0) - H(x - \xi_{n+1})), \tag{3}$$

where

$$\hat{F}_{\tilde{X}}(x) = \begin{cases} F_{\tilde{X}}(\xi_0) \quad (> 0), & x < \xi_0, \\ F_{\tilde{X}}(x), & \xi_0 \le x < \xi_{n+1}, \\ \lim\limits_{\varepsilon \to 0} \left(F_{\tilde{X}}(\xi_{n+1} - |\varepsilon|)\right) \quad (< 1), & x \ge \xi_{n+1}. \end{cases} \tag{4}$$

This representation of $F_{\tilde{X}}$ enables us also to simplify $w^- \circ F_{\tilde{X}}$ and $w^+ \circ (1 - F_{\tilde{X}})$ according to

$$w^{-}\left(F_{\tilde{X}}(x)\right) = H(x - \xi_{n+1}) + w^{-}\left(\hat{F}_{\tilde{X}}(x)\right) \cdot (H(x - \xi_0) - H(x - \xi_{n+1})) \tag{5}$$

and

$$w^{+}\left(1 - F_{\tilde{X}}(x)\right) = (1 - H(x - \xi_0)) + w^{+}\left(1 - \hat{F}_{\tilde{X}}(x)\right) \cdot (H(x - \xi_0) - H(x - \xi_{n+1})) \tag{6}$$

because $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$. The advantage of these representations of $w^- \circ F_{\tilde{X}}$ and $w^+ \circ (1 - F_{\tilde{X}})$ is the differentiability of $w^- \circ \hat{F}_{\tilde{X}}$ and $w^+ \circ (1 - \hat{F}_{\tilde{X}})$ at $x = \xi_0$ and $x = \xi_{n+1}$ if $\xi_0, \xi_{n+1} \in \Re$. However, against the background of (1), (5), and (6), we need to determine the derivative of the Heaviside function (under the integral).

[4] See, e.g., Köbberling and Wakker (2005) for a suggestion of a value function that fulfills the desired assumptions.

[5] Because the development of CPT for piecewise continuous distributions is the centerpiece of the present article, the derivation is presented in detail in this section and is not placed in the appendix.

[6] The Heaviside function is a non-continuous function whose value is zero for negative arguments and one for positive arguments. The value of the function at zero is ambiguous. Usually, the value is defined as $H(0) = 0$, $H(0) = 0.5$, or $H(0) = 1$. Within the framework of the present paper, we set $H(0) = 1$.