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Time-consistent mean–variance asset–liability management with random coefficients

Jiaqin Wei [a], Tianxiao Wang [b,*]

[a] School of Statistics, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China
[b] School of Mathematics, Sichuan University, Chengdu 610065, China

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ABSTRACT

In this paper, we aim to find a time-consistent open-loop equilibrium strategy for the asset–liability management problem under mean–variance criterion. The financial market consists of a bank account and m stocks whose prices are modeled by geometric Brownian motions. The liability of the investor is uncontrollable and modeled by another geometric Brownian motion which is correlated to the stock prices. First, we provide a sufficient condition for the equilibrium strategy, which involves a system of FBSDEs. Second, by solving these FBSDEs, we obtain an equilibrium strategy in a linear feedback form of the surplus and the liability. Finally, we consider a Markovian case where the interest rate is given by the Vasiček model.

1. Introduction

The pioneer work of Markowitz (1952) studies the portfolio selection under the well-known mean–variance criterion and derives the analytical expression of the mean–variance efficient frontier in the single-period model. This seminal work has become the foundation of modern portfolio theory and has stimulated numerous extensions.

On the one hand, some researchers focus on studying the dynamic mean–variance portfolio selection problem. Samuelson (1969) considers a discrete-time multi-period model. More recently, by embedding the original problem into a stochastic linear–quadratic (LQ) control problem, Li and Ng (2000) and Zhou and Li (2000) extend Markowitz's work to a multi-period model and a continuous-time model, respectively. Lim (2004) studies the quadratic hedging and mean–variance portfolio selection in an incomplete market. On the other hand, there are some works that consider a generalized financial market. An important and popular subject is the asset and liability management problem, which is concerned with the selection of portfolio while taking the liabilities of investors into account. More specifically, the surplus defined as the difference between the asset and liability is considered in the asset and liability management problem.

Since it was proposed by Sharpe and Tint (1990) in a single-period model, there is an increasing number of interests in the asset–liability management (ALM, for short) under the mean–variance criteria. Keel and Müller (1995) studies the portfolio choice with liabilities, and shows that liabilities affect the efficient frontier. By the embedding technique of Li and Ng (2000), Leippold et al. (2004) derive an analytical optimal policy and efficient frontier for the multi-period ALM problem. The mean–variance ALM in a continuous-time model is investigated by Chiu and Li (2006) from the view of stochastic LQ control problem, where both the optimal strategy and the efficient frontier are obtained. Furthermore, in a regime-switching framework, Chen et al. (2008) and Chen and Yang (2011) investigate the mean–variance ALM in the continuous-time model and multi-period model, respectively. All of these papers assume that the liabilities are not controllable, which is the main difference between the Markowitz's problem and the ALM.

It is well acknowledged that due to the existence of a non-linear function of the expectation in the objective functional, the dynamic mean–variance portfolio selection problem is time inconsistent in the sense that the Bellman optimality principle does not hold. Intuitively, the optimal strategy obtained for the initial time may not be optimal for some latter time. This is the so-called pre-committed strategy which is only optimal for the initial time. Observe that only pre-committed strategies are considered in all the aforementioned references.

Strotz (1955) first studies the time-inconsistent problem by the game theoretic approach. More specifically, Strotz views the time inconsistent problem as a non-cooperative game, in which there is a player at each time t (the player can be viewed as the future

* Corresponding author.
E-mail addresses: jqwei@stat.ecnu.edu.cn (J. Wei), wtxiao2014@scu.edu.cn (T. Wang).

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incarnation of the decision-maker at time t). Then he seeks a subgame perfect Nash equilibrium strategy which is time-consistent. Recently, there is an increasing amount of attentions in the time inconsistent control problem due to the practical applications in the economics and finance. In Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008) which consider the optimal consumption and investment problems with hyperbolic discounting, the precise definition of the equilibrium solution in continuous time is introduced for the first time. Following their ideas, Björk and Murgoci (2010) investigate a class of time-inconsistent control problem in a general Markovian framework. They derive a system of extended HJB equation and prove the associated verification theorem. Björk et al. (2014) studies the Markowitz's problem with state-dependent risk aversion by the extended HJB equation approach and shows that the equilibrium control is dependent on the current state. Wei et al. (2013) considers a regime-switching model with state dependent risk aversion and obtains the equilibrium strategy for the mean–variance asset–liability management problem by the previous extended HJB equation as well.

In Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and following-up papers, the equilibrium control is defined within the class of closed-loop (i.e. feedback) controls. In the particular LQ framework, Hu et al. (2012) defines the equilibrium control within the class of open-loop controls and derives a general sufficient condition for equilibriums through a system of forward–backward stochastic differential equations (FBSDEs). As an application, they deal with the Markowitz's problem with state-dependent risk aversion and random coefficients. However, the interest rate in their model is assumed to be deterministic. More recently, Hu et al. (2015) continues to discuss the uniqueness of open-loop equilibrium strategies. At this moment, it is worthy of mentioning the work of Yong (2015), where both the open-loop and close-loop equilibrium strategies of linear–quadratic optimal control problems for mean-field stochastic differential equations are studied.

In this paper, we study the mean–variance ALM problem in continuous-time setting. Similar to Hu et al. (2012), our object is to find open-loop equilibrium strategy as well. To capture the fluctuations of the parameters of the financial market and the liability in the long-term portfolio selection problem, we allow all the parameters (including the interest rate) to be random. First, we establish a sufficient condition of equilibrium strategy via a system of FBSDEs. The method of the proof is different from that in Hu et al. (2012). Second, by introducing a series of backward stochastic differential equations (BSDEs), we construct a solution to the previous system of FBSDEs and a strategy in linear feedback form of surplus and liability. Under a technical condition (see Proposition 3.9), we verify that the strategy is indeed an equilibrium strategy for the mean–variance ALM problem. Finally, we consider a Markovian case where the interest rate is given by the Vasiček model. In this case, by solving a series of partial differential equations (PDEs), we obtain an equilibrium strategy and the corresponding efficient frontier in the closed-form. Some comparisons between the open-loop and closed-loop equilibrium strategies are given. It is shown that the randomness of the interest rate leads to differences between these two kinds of strategies. We also present a numerical example to show how the volatility of the interest rate affects the efficient frontier in the case without liability.

The remainder of this paper is organized as follows. In Section 2, we give the formulation of the mean–variance ALM problem and introduce the definition of the equilibrium strategy. In Section 3, we present a sufficient condition for equilibrium strategy, and then give an equilibrium strategy and the corresponding value function. In Section 4, we study an example with Vasiček interest rate model. Section 5 concludes the paper. In the Appendices we collect some lengthy proofs.

2. The model

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a fixed complete probability space on which an m-dimensional standard Brownian motion $\boldsymbol{W}(\cdot) \equiv (W_1(\cdot), \dots, W_m(\cdot))^\top$ and an n-dimensional Brownian motion $\boldsymbol{B}(\cdot) \equiv (B_1(\cdot), \dots, B_n(\cdot))^\top$ are defined. Let $T > 0$ be the fixed and finite time horizon, $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ be the augmented filtration generated by $(\boldsymbol{W}(\cdot), \boldsymbol{B}(\cdot))$ and $\mathbb{F}^{\boldsymbol{W}} := \left\{\mathcal{F}_t^{\boldsymbol{W}}\right\}_{t\in[0,T]}$ be the augmented filtration generated by $\boldsymbol{W}(\cdot)$. We shall use $\boldsymbol{W}(\cdot)$ and $\boldsymbol{B}(\cdot)$, which are assumed to be independent of each other, to describe the risks of the financial market and the other risks faced by the investor (e.g., the claim risks, if the investor is an insurer), respectively.

For $p \geq 1$, $\boldsymbol{H} := \mathbb{R}^n, \mathbb{R}^{n\times m}$, etc. and $0 \leq s \leq t \leq T$, we define

$$\begin{aligned}
L^p_{\mathcal{F}_t}(\Omega; \boldsymbol{H}) &:= \Big\{ \boldsymbol{X} : \Omega \to \boldsymbol{H} \;\Big|\; \boldsymbol{X}(\cdot) \text{ is } \mathcal{F}_t\text{-measurable}, \\
&\qquad \mathsf{E}\left[|\boldsymbol{X}|^p\right] < \infty \Big\}, \\
L^p_{\mathbb{F}}(s, t; \boldsymbol{H}) &:= \Big\{ \boldsymbol{X} : [s,t]\times\Omega \to \boldsymbol{H} \;\Big|\; \boldsymbol{X}(\cdot) \text{ is } \mathbb{F}\text{-adapted}, \\
&\qquad \mathsf{E}\left[\int_s^t |\boldsymbol{X}(v)|^p \mathrm{d}v\right] < \infty \Big\}, \\
L^p_{\mathbb{F}}\left(\Omega; L^2(s, t; \boldsymbol{H})\right) &:= \Big\{ \boldsymbol{X} : [s,t]\times\Omega \to \boldsymbol{H} \;\Big|\; \boldsymbol{X}(\cdot) \text{ is } \mathbb{F}\text{-adapted}, \\
&\qquad \mathsf{E}\left[\left(\int_s^t |\boldsymbol{X}(v)|^2 \mathrm{d}v\right)^p\right] < \infty \Big\}, \\
L^p_{\mathbb{F}}(\Omega; C([s,t]; \boldsymbol{H})) &:= \Big\{ \boldsymbol{X} : [s,t]\times\Omega \to \boldsymbol{H} \;\Big|\; \boldsymbol{X}(\cdot) \text{ is } \mathbb{F}\text{-adapted}, \\
&\qquad \text{has continuous paths}, \\
&\qquad \text{and } \mathsf{E}\left[\sup_{v\in[s,t]} |\boldsymbol{X}(v)|^p\right] < \infty \Big\}.
\end{aligned}$$

In what follows, unless specified otherwise, we adopt boldface letters to denote matrices and vectors, and the transpose of a matrix or vector $\boldsymbol{M}$ is denoted by $\boldsymbol{M}^\top$. Also, we denote by M_{ij} (resp. M_i) the (i,j)-element (resp. the ith element) of the matrix (resp. vector) $\boldsymbol{M}$.

We consider a financial market consisting of a bank account and m stocks within the time horizon $[0, T]$. The bank account $S_0(\cdot)$ is governed by

$$\begin{cases} \mathrm{d}S_0(s) = r(s)S_0(s)\mathrm{d}s, & s\in[0,T], \\ S_0(0) = s_0 > 0, \end{cases} \tag{2.1}$$

where $r(\cdot) > 0$ is the interest rate. For $i = 1, 2, \dots, m$, the price of the ith stock $S_i(\cdot)$ is given by

$$\begin{cases} \mathrm{d}S_i(s) = S_i(s)\left[\mu_i(s)\mathrm{d}s + \displaystyle\sum_{j=1}^m \sigma_{ij}(s)\mathrm{d}W_j(s)\right], & s\in[0,T], \\ S_i(0) = s_i > 0, \end{cases} \tag{2.2}$$

where $\mu_i(\cdot)$ is the expected return rate of the ith risky asset and $\sigma_{ij}(\cdot)$ is the corresponding volatility rate. In the following, $r(\cdot)$ is a bounded continuous $\mathbb{F}^{\boldsymbol{W}}$-adapted process, $\boldsymbol{\mu}(\cdot) := (\mu_1(\cdot), \dots, \mu_m(\cdot))^\top$ and $\boldsymbol{\sigma}(\cdot) := (\sigma_{ij}(\cdot))_{1\leq i,j\leq m}$ are bounded continuous $\mathbb{F}$-adapted processes such that $\mu_i(\cdot) > r(\cdot)$, $\boldsymbol{\sigma}(\cdot)\boldsymbol{\sigma}^\top(\cdot) \geq \delta \boldsymbol{I}_{m\times m}$, with constant $\delta > 0$.

We denote by $L(\cdot)$ the liability of the investor, the dynamics of which is shown as

$$\begin{cases} \mathrm{d}L(s) = L(s)\left[\alpha(s)\mathrm{d}s + \boldsymbol{\beta}^\top(s)\mathrm{d}\boldsymbol{W}(s)\right. \\ \qquad \left. + \boldsymbol{b}^\top(s)\mathrm{d}\boldsymbol{B}(s)\right], & 0 \leq s \leq T, \\ L(0) = l_0 > 0, \end{cases} \tag{2.3}$$