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Optimal insurance design with a bonus[☆]

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ABSTRACT

This paper investigates an insurance design problem, in which a bonus will be given to the insured if no claim has been made during the whole lifetime of the contract, for an expected utility insured. In this problem, the insured has to consider the so-called optimal *action* rather than the contracted compensation (or indemnity) due to the existence of the bonus. For any pre-agreed bonus, the optimal insurance contract is given explicitly and shown to be either the full coverage contract when the insured pays high enough premium, or a deductible one otherwise. The optimal contract and bonus are also derived explicitly if the insured is allowed to choose both of them. The contract turns out to be of either zero reward or zero deductible. In all cases, the optimal contracts are universal, that is, they do not depend on the specific form of the utility of the insured. A numerical example is also provided to illustrate the main theoretical results of the paper.

1. Introduction

Risk sharing, also known as "risk distribution", is a method of managing or reducing risk exposure by spreading the burden of loss among each member of a group based on a predetermined formula. It can be mathematically formulated as a multi-objective optimization problem in which a Pareto optimality is sought with respect to each member's risk preference.

In the context of insurance, the primary risk sharing problem is the designing of an insurance contract that achieves Pareto optimal for the (typically two) involved parties: the insurer and the insured. Specifically, given an upfront premium that the insured pays the insurer, the classical insurance design problem is to determine the (contracted) amount of loss $I(X)$ covered by the insurer – called compensation or indemnity – for a random loss X. In order to let the insurer have sufficient incentive to offer the contract, on top of the actuarial value of the contracted compensation, the premium should also cover a safety loading in addition—this is the so-called participation constraint of the insurer in the literature. Once the loss occurs, the insured will claim it and ask the insurer to cover the contracted amount of loss $I(X)$. Not only in theory but also in practice, the optimal designing of insurance contract is fundamentally important.

In the designing of an insurance contract, the insured's and the insurer's risk preferences manifestly play the key role. To model them, the classical expected utility theory (EUT), non-EUTs or mixed risk preferences have been considered in the insurance literature. The EUT models are vast, and in these models the insurer is often assumed to be risk-neutral while the insured is assumed to be risk-averse; see, e.g., Arrow (1963, 1974), Raviv (1979), and Gollier and Schlesinger (1996). The optimal compensation usually turns out to be a deductible one in which the insurer covers the amount of loss exceeding a deductible level. Such theoretical result is consistent with most of the insurance contracts available in practice. However, the EUT has received many criticisms for its failure in describing numerous human behaviors or explaining experimental observations (see, e.g., Allais, 1953; Mehra and Prescott, 1985), so that many non-EUTs have been introduced to overcome the drawback of the EUT. For instance, Quiggin (1982) proposed the rank-dependent utility theory (RDUT); Tversky and Kahneman (1992) proposed the cumulative prospect theory (CPT) (see Barberis, 2013 for an excellent survey). A number of papers have already studied insurance contract design problems in the RDUT or CPT frameworks; see, e.g., Chateauneuf et al. (2000), Carlier and Dana (2005), Dana and Scarsini (2007), Bernard et al. (2015) and Xu et al. (2016). At the meanwhile, other risk preferences including VaR and CTE have also been widely considered, see, e.g., Cai and Tan (2007) and Cai et al. (2008).

At the meanwhile, in many standard insurance contracts today, the bonus–malus system is in place. The term bonus–malus is Latin for good–bad. This system records the insured's history (including both good and bad events) to determine her premium today. For instance, when the insured made a claim due to a car accident, her premium for the next contract may increase. This paper investigates an insurance design problem in which a bonus will be

☆ This work has been done when Li was visiting Xu at The Hong Kong Polytechnic University.

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given to the insured if no claim has been made during the whole lifetime of the contract. This is a bonus–malus system problem. In such a system, the insured will compare the compensation with the potential bonus to be awarded by hiding her losses. This makes her to consider the so-called optimal *action* rather than the contracted compensation to optimize her risk preference. The problem is considered in the classical expected utility framework in this paper. The explicit contract is derived for each pre-agreed reward, either being the full coverage contract when the insured pays high enough premium, or being a deductible one otherwise. The optimal compensation and bonus are also derived when the insured is allowed to choose both of them. In all cases, the optimal contracts are universal, that is, they do not depend on the specific form of the utility of the insured.

The remainder of this paper is organized as follows. We mathematically formulate the problem in Section 2. We derive the optimal contract for any pre-agreed bonus in Section 3 and provide a numerical example to illustrate the theoretical results. Section 4 is devoted to the study of optimal personalized contract, i.e., a contract that allows the insured to choose both the compensation and bonus. We conclude the paper in Section 5.

2. Model formulation

In this section, we formulate an optimal insurance design problem in which a bonus will be given to the insured if no claim has been made during the whole lifetime of the contract.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. An insured, endowed with an initial wealth $\varpi > 0$, faces a random loss $X \geqslant 0$. She chooses an insurance contract, by paying a premium π to the insurer in return for a *compensation* (or *indemnity*) in the case of a loss, to protect herself from the loss. This contracted compensation is a function of the loss, denoted by $I(\cdot)$. In this paper, the compensation is also called *contract* as it clearly determines the essentials of the insurance contract. In our model, the insured will be paid a pre-agreed bonus θ if no claim has been made during the whole lifetime of the contract. It is this bonus feature that distinguishes our model from those in insurance design literature. Intuitively speaking, when facing a loss, the insured shall compare the instant loss with the potential bonus to decide whether to claim the loss. Such consideration leads her to take actions deviating from the contracted compensation $I(\cdot)$. We assume the insured will act as a function of the loss, denoted by $A(\cdot)$, called an *action*. We should note that any action is a consequence of some contracted compensation. In the absence of bonus, the action and the compensation are the same. In contrast, in our model, the insured will receive a bonus θ if no claim has been made, therefore we have the *realized compensation*

$$C(X) = \begin{cases} A(X), & A(X) > 0; \\ \theta, & A(X) = 0; \end{cases} = A(X) + \theta \mathbb{1}_{A(X)=0}.$$

This is the real amount that the insured will receive from the insurer. Its right hand side highlights the bonus feather of the model. As usual, here and hereafter, we use $\mathbb{1}_S$ to denote the indicator function of a sentence S, thus $\mathbb{1}_S$ equals 1 when S is true and equals zero otherwise.

The insurer designs an insurance contract from the insured's point of view. For a potential loss X, the insured aims to choose an insurance contract (and hence the corresponding action) that provides the best tradeoff between the premium and the realized compensation based on her risk preference. In this paper, we consider an expected utility preference insured whose utility is $u(\cdot)$ mapping $\mathbb{R}$ to $\mathbb{R}^+$, so that her objective is to maximize

$$\begin{aligned}&\mathbf{E}\big[u\big(\varpi - \pi - X + C(X)\big)\big]\\ &\quad = \mathbf{E}\big[u\big(\varpi - \pi - X + A(X) + \theta \mathbb{1}_{A(X)=0}\big)\big].\end{aligned}$$

On the other hand, the insurer is risk-neutral and the cost of offering the contract is proportional to the expectation of the realized compensation, so the premium to be charged for a realized compensation should satisfy the participation constraint

$$\pi \geqslant (1+\rho)\mathbf{E}[C(X)] = (1+\rho)\mathbf{E}\big[A(X) + \theta \mathbb{1}_{A(X)=0}\big],$$

where the constant $\rho \geqslant 0$ is the *safety loading coefficient* of the insurer.

It is natural to require any contracted compensation to satisfy

$$I(0) = 0, \quad 0 \leqslant I(x) \leqslant x, \quad \forall\, x \geqslant 0,$$

a constraint that has been imposed in the most insurance design literature. In our framework, the action $A(\cdot)$ may be different from the contracted compensation $I(\cdot)$. But clearly in no situation, the insured can claim more than $I(\cdot)$. Hence it is natural to require

$$0 \leqslant A(x) \leqslant I(x), \quad \forall\, x \geqslant 0.$$

On the other hand, the insured will choose the best realized compensation (rather than the contracted compensation) in the presence of bonus, so the above constraint can be relaxed to

$$A(0) = 0, \quad 0 \leqslant A(x) \leqslant x, \quad \forall\, x \geqslant 0. \tag{2.1}$$

Once the best action has been found, one should recover a contract (namely contracted compensation) that will lead to this best action. Meanwhile, we require the action to be globally increasing. Economically speaking, this means the insured's compensation is comonotone increasing with respect to the loss, asking more when a bigger loss occurs. Mathematically speaking, we require

$$A(y) \leqslant A(x), \quad \forall\, y \leqslant x.$$

We can now formulate our bonus–malus system insurance design problem with a (pre-agreed) bonus $\theta \geqslant 0$ as

$$\begin{aligned}&\max_{A(\cdot)\in\mathcal{A}} \quad \mathbf{E}\big[u\left(\varpi - \pi - X + A(X) + \theta \mathbb{1}_{A(X)=0}\right)\big] \\ &\text{subject to} \quad (1+\rho)\mathbf{E}\big[A(X) + \theta \mathbb{1}_{A(X)=0}\big] \leqslant \pi,\end{aligned} \tag{2.2}$$

where the set of admissible actions is given by

$$\mathcal{A} = \{A(\cdot) : A(0) = 0, \quad A(y) \leqslant A(x) \leqslant x, \quad \forall\, 0 \leqslant y \leqslant x\}. \tag{2.3}$$

We denote by $F(\cdot)$ the probability distribution function of the potential loss X. For simplicity we assume that $F(\cdot)$ is strictly increasing and differentiable on $(0, +\infty)$ so that X has no atoms on $(0, +\infty)$. This assumption however allows the loss X to have a mass at 0, which is of course the most common case in insurance practice. Since $X \geqslant 0$, we have $F(0-) = \mathbf{P}(X < 0) = 0$. In addition, we also assume that the loss X has a finite expectation so that $\int_{[0,\infty)} x\,\mathrm{d}F(x) = \mathbf{E}[X] < \infty$. All these assumptions are technical and can be relaxed to more general cases without too much difficulties; this, however, is not the pursuit of the present paper.

Remark 2.1. In contrast to Xu et al. (2016), we *donot* require both the action and the real retention to be globally increasing. Different from Bernard et al. (2015) where a severe problem of moral hazard has arisen as their contract is not increasing with respect to the loss due to lack of the requirement, our optimal contract eventually turns out to satisfy the requirement automatically. The reason behind it is that we consider an EU preference insured rather than a RDUT one as in Bernard et al. (2015). The moral hazard problem must be carefully treated if one considers a RDUT preference insured.