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Analytical valuation and hedging of variable annuity guaranteed lifetime withdrawal benefits

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ABSTRACT

Variable annuity is a retirement planning product that allows policyholders to invest their premiums in equity funds. In addition to the participation in equity investments, the majority of variable annuity products in today's market offer various types of investment guarantees, protecting policyholders from the downside risk of their investments. One of the most popular investment guarantees is known as the guaranteed lifetime withdrawal benefit (GLWB). In current market practice, the development of hedging portfolios for such a product relies heavily on Monte Carlo simulations, as there were no known closed-form formulas available in the existing actuarial literature. In this paper, we show that such analytical solutions can in fact be determined for the risk-neutral valuation and delta-hedging of the plain-vanilla GLWB. As we demonstrate by numerical examples, this approach drastically reduces run time as compared to Monte Carlo simulations. The paper also presents a novel technique of fitting exponential sums to a mortality density function, which is numerically more efficient and accurate than the existing methods in the literature.

1. Introduction

Variable annuities (VAs) are life insurance contracts that offer policyholders participation in equity investments. They provide life contingent benefits like traditional life insurance or annuities, while allowing policyholders to reap benefits of financial returns on their premiums. Variable annuities were introduced in the United States in the 1950s with the earliest products made available by the Teachers Insurance and Annuity Association of America (TIAA)—College Retirement Equities Fund (CREF). They were created to provide incomes with investment returns to retired professors. Nowadays variable annuities have become popular products available through direct purchase or through tax-sheltered retirement savings plans such as IRAs, 401(k)s, 403(b)s, etc. Various forms of variable annuities are also sold in many other markets, such as in Japan (cf. Zhang, 2006) and in Europe (cf. O'Malley, 2007). According to Life Insurance and Market Research Association (LIMRA)'s annuity sales survey LIMRA (2016), sales of variable annuities amounted to $133 billions in 2015 in the US. According to the joint study by the Society of Actuaries and LIMRA (cf. Drinkwater et al., 2014), the GLWB is the most popular type, accounting for three quarters of the sales, of all guaranteed living benefits, which include the GLWB, guaranteed minimum withdrawal benefit (GMWB), guaranteed minimum income benefit (GMIB) and guaranteed minimum accumulation benefit (GMAB).

Let us briefly describe the design of a plain vanilla GLWB. As with any VA base contract, policyholders are offered a variety of equity funds to invest consisting of different combinations of equities and bonds. Upon selection, future financial returns on the policyholders' investment accounts are linked to equity funds of their choices. The GLWB is a rider that policyholders can add to their base contracts. In essence, the GLWB rider guarantees policyholders' incomes for lifetime without having to annuitize (convert to an immediate annuity). It allows policyholders to take withdrawals at no additional cost up to a maximum amount per year for lifetime, even if their investment accounts are depleted. In addition, at the time of a policyholder's death, the remaining balance in the account, if any, will be returned to the policyholder's beneficiary. Any withdrawal amount beyond the maximum amount is also allowed but subject to a penalty. To compensate for the insurer's expenses and cost of guarantees, charges are made as a fixed percentage of policyholders' investment and deducted from their accounts, typically on a daily basis.

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Although withdrawal activities can vary due to policyholders' behaviors, SOA and LIMRA's 2015 experience study (cf. Drinkwater et al., 2015, p. 20) shows that "the majority of owners take withdrawals through systematic withdrawal plans (SWPs)... When owners use SWPs, they are likely to make withdrawals within the maximum amount allowed in their contracts". This phenomenon has been observed consistently for many years by SOA and LIMRA's annual experience studies. The 2015 experience study shows that "79% of owners who took withdrawals in 2013 withdrew income that was below or close to the maximum amount calculated—up to 110%" and roughly 55% of policyholders who took withdrawals withdrew income between 90% and 110% of the maximum amount. Drinkwater et al. (2015, p. 24) also report that "owners rarely add premium after the second year of owning a GLWB". Therefore, in this paper, we make simplifying assumptions which are satisfied in the majority of cases in practice. (1) The policyholder under consideration starts taking withdrawals immediately at the time of valuation, although the results in this paper can be easily extended to address the accumulation of investment account in a waiting period until the first withdrawal. (2) The policyholder takes withdrawals at the penalty-free maximum amount every year. (3) There is only an initial purchase payment (premium) at the start of the contract and no additional deposits are considered.

Consider a numerical example for the illustration of GLWB benefits. Suppose identical variable annuity contracts with the same GLWB rider are sold to two policyholders both of age 65 at policy issue. Each of them makes an initial purchase payment of \$100 and their accounts are linked to the same equity index. They take withdrawals at the maximum amount, which is 7% of the initial purchase payment per year. Note that the two contracts may end at different points of time due to the uncertainty with each policyholder's time of death. Suppose that the equity fund performs well at first, resulting that both policyholders' investment accounts accumulate to \$110 at the end of the second year. Policyholder A dies at the end of the second year, denoted by $T^A_{65} = 2$ in Fig. 1. He and his beneficiary would have received a total of \$124 by then, including the withdrawals from the first two years $\$7 \times 2 = \$14$ and the remaining balance at the time of death \$110. In this case, the insurer collects rider charges and other fees for two years but makes no liability payment as policyholder A's account remains positive throughout his lifetime. Suppose that the market takes a sharp down turn afterwards, which causes the surviving policyholder B's account value to deplete in the ninth year. Even though the investment account has no money left, policyholder B continues to receive the guaranteed withdrawals of \$7 per year. Policyholder B dies at the end of the twentieth year, $T^B_{65} = 20$, so policyholder B and her beneficiary would have been paid $\$7 \times 20 = \$140$ in total. Note, however, in this case, not all 20 installments of withdrawals would be considered as GLWB benefit, since $\$7 \times 8 = \$56$ is taken out of policyholder B's own investment account. After the account balance hits zero in the ninth year, the rest $7 \times 12 = \$84$ is the actual out-of-pocket cost of the GLWB liability for the insurer.

Although the modeling and valuation of guaranteed withdrawal benefits started in the insurance industry since their introduction to the market in 2004, Milevsky and Salisbury (2006) were among the first to lay out the theoretical framework in the actuarial literature. The authors introduced a continuous time model to price a GMWB from the policyholder's perspective and used numerical partial differential equation (PDE) methods for solutions. Their work was extended to more sophisticated setting by many authors, including Chen and Forsyth (2008), Peng et al. (2012), Dai et al. (2008) and Bacinello et al. (2016), etc. A recent work by Feng and Volkmer (2016b) provided an analytical solution to the risk-neutral valuation of the GMWB from both the perspective of a policyholder and that of an insurer. As the GLWB can be viewed as an extension of the GMWB, many researchers employed PDE and Monte Carlo simulation techniques to value and analyze the GLWB, such as Piscopo and Haberman (2011), Fung et al. (2014) and Huang et al. (2014), etc. This work extends the analytical method developed in Feng and Volkmer (2016b) to a plain vanilla GLWB in the same framework as in Fung et al. (2014). One might question whether it is worthwhile looking for analytical solutions while numerical PDEs and simulation methods are available to handle much more complex designs. Here are a number of reasons why analytical solutions, if available, are always preferred.

1. Analytical solutions provide benchmarks against which the accuracy of PDE and simulation algorithms can be tested. This is often an overlooked step in industrial practice. Practitioners often check if their Monte Carlo estimates reach a certain degree of accuracy by increasing their sample sizes, thereby checking the consistency of results. However, such convergence tests only work if the statistics are unbiased or consistent estimators. Simulation methods cannot easily detect inherent biases that may exist due to various combinations of approximation techniques.
2. Analytical solutions can be used to obtain cost effective approximations of Greeks. Risk neutral valuation is often required for determining sensitivity measures for hedging. As the Greeks are often approximated by difference quotients, modest sampling errors in risk neutral valuations can lead to large relative errors in the estimation of Greeks. Such an example can be seen in Table 8 of Section 3.2. In the financial industry, the use of the so-called "volatility smile" in Black–Scholes formulas for pricing and hedging has been widely known as "the wrong number in the wrong formula to get the right price". In other words, a common practice is often to find a comprise between the complexity of the underlying model and the efficiency of results to be delivered. In the same spirit, even though analytical solutions may be used on idealistic assumptions, they provide highly efficient and low cost approximations for otherwise difficult but precise solutions.

The paper also dedicates a significant portion to new techniques for fitting sums of exponentials to probability density functions. For comparison, we provide a self-contained review of various statistical and analytical methods for approximating mortality density functions by exponential sums, some of which are introduced for the first time to the actuarial literature. The approximation based on Hankel matrix appears much more efficient at the same level of accuracy than known methods in actuarial literature such as orthogonal polynomials. Hence we shall only apply the former in numerical examples in Section 3. To avoid any digression from the main theme of pricing and hedging of the GLWB rider, we relegate the results on fitting exponential sums to Appendix. Nevertheless, it should be pointed out that Appendix B could be of interest on its own and have boarder actuarial and financial applications than pricing and hedging in this paper.

The rest of the paper is organized as follows. In Section 2, the risk-neutral valuation model for the GLWB rider is introduced from both a policyholder's and an insurer's perspectives. We provide analytical solutions to both risk-neutral values of the GLWB liabilities and the corresponding deltas. We develop in Section 3 a number of numerical examples to compare the proposed analytical methods with traditional Monte Carlo methods.

2. Models

2.1. Equity-linking mechanism

Suppose that a policyholder's investment is linked to a single equity index, which is driven by a geometric Brownian motion and