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Multi-period risk sharing under financial fairness

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ABSTRACT

We work with a multi-period system where a finite number of agents need to share multiple monetary risks. We look for the solutions that are both Pareto efficient utility-wise and financially fair value-wise. A buffer enables the inter-temporal capital transfer. Expected utility is used to evaluate the utility, and a risk-neutral measure is essential for determining the risk sharing rules. It can be shown that in the model setting there always exists a unique risk sharing rule that is both Pareto efficient and financially fair. An iterative algorithm is introduced to calculate this rule numerically.

1. Introduction

This paper explores the intertemporal risk sharing in a multi-period setting under the notion of *Pareto efficiency and financial fairness* (PEFF). Pareto efficiency means that the utility of nobody can be improved without hurting the utility of some others, while financial fairness indicates that the market values of the risk positions before and after risk sharing should be equal. A risk-sharing system with respect to monetary uncertainties – the stochastic returns from the financial market, for instance – can be viewed as a financial contract. On the one hand, Pareto efficiency is fundamental in risk-sharing systems, while on the other hand financial fairness is important in the design of financial contracts.

The model is motivated and abstracted from systems that allow for intertemporal risk sharing. One example is the collective defined-contribution pension systems which can be viewed as a multilateral financial contract among both current and future cohorts. The possibility of intertemporal risk sharing with respect to investment risk is due to the incompleteness of the market, i.e. the inability of generations to be exposed to risks outside their own (mature) lifespan. A risk-sharing system tries to partly fix this problem by allowing later generations to take risks before they become participants. Risk sharing shall result in welfare gains to the generations; meanwhile, the pension contract should also be fair from a valuation perspective. Another example is the reinsurance market, in which insurance companies reallocate the risks by way of reinsurance contracts among themselves. A multi-period contract is appropriate for dealing with long-term risks, or simply when companies agree to make multi-period arrangements. A similar example is the design of structured derivatives, for instance, the practice of tranching. In these examples, Pareto efficiency is pertinent for designing the optimal allocation of risks, while financial fairness guarantees that the contract is fairly priced.

The characterization of Pareto efficient solutions in a single-period setting is well studied in a lot of papers, which date back to the 1960s with the focus mainly on the field of insurance. For instance, Borch (1962) gives a characterization of the Pareto efficient solutions under the situation where expected utility is used to describe the agents' risk preferences, and later DuMouchel (1968) gives proof to these results. Similar work also includes Raviv (1979) which takes into consideration the existence of market frictions. The fairness criterion is first considered alongside the Pareto efficiency by, amongst others, Gale (1977), Bühlmann and Jewell (1979) and Balasko (1979) in different settings. In these literature, the risk sharing is built over both a utility basis and a valuation basis.

The risk sharing problem in a multi-period setting is investigated by Barrieu and Scandolo (2008) in a general setting; they talk about risk exchanges between two agents over more than one

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period without taking into consideration any fairness conditions. Other work has been mainly focused on the design of pension systems and the space of intergenerational risk sharing, where risk redistribution can be organized among both the existing and future cohorts. Pareto-efficient risk sharing can be achieved by maximizing the aggregate expected utility of all generations in the situation where a social planner is present (e.g. Gordon and Varian, 1988; Gollier, 2008; Bovenberg and Mehlkopf, 2014) or by looking for an equilibrium (see Ball and Mankiw, 2007; Krueger and Kubler, 2006). Financial fairness has been considered by Cui et al. (2011); however, the valuation approach is only used to check afterwards whether the distribution rule is fair for the participants. Kleinow and Schumacher (in press) analyze the pension system with conditional indexation from the perspective of market value; they investigate whether the pension contract is financially fair for existing and incoming cohorts as well as the sponsor. Risk-neutral valuation becomes essential in Bovenberg and Mehlkopf (2014) to determine a unique risk sharing solution by setting the ex-ante market values of the intergenerational transfers to zero.

This paper explores the Pareto efficient and financially fair risk sharing in a multi-period environment. Expected utility is adopted to evaluate the welfare, and a risk-neutral measure works for the valuation purpose. We shall show the existence and uniqueness of the PEFF solution, and give a numerical algorithm to find it. This can be seen as a direct generalization of the research by Pazdera et al. (2016), which explores the Pareto efficient and financially fair risk-sharing rule in a single-period case. Compared to Barrieu and Scandolo (2008), we restrict ourselves to the case of expected utility as the preference functional, and risk-neutral valuation is built into the system to determine the uniqueness of the solution. Different from Bovenberg and Mehlkopf (2014), no parameterization on the risk-sharing rules is needed here; the rules are determined totally under the notion of PEFF. Mathematically, our results resemble the famous consumption–savings model for intertemporal substitution to some extent. The intertemporal balance equation, as we call it, has a close relationship with the Euler equation in the intertemporal substitution theory; see Hall (1978). The main difference is that the model here introduces no subjective discount factor for impatience. The characterization of Pareto efficiency leads to a weighted optimization problem where the weights are unknowns to be determined uniquely by the financial fairness constraints, making use of a risk-neutral measure.

The rest of the paper is structured as follows. The model setting is set up in Section 2 and we formulate the problem of finding PEFF solutions mathematically. Next we establish the existence and uniqueness of the solution in Section 5. Explicit solution exists when we assume exponential utility functions to all the agents and deterministic asset returns; other than that, there appears to be no hope for an explicit solution in general. We then develop an iterative algorithm to numerically find the solution. The case of the explicit solution is dealt with in Section 7; besides, we also give a simple example where the numerical algorithm is implemented. Some remarks will conclude the paper in the end.

2. Model framework

We assume a finite discrete-time system in which a finite number of agents gather to share their risks. As a result of the risk sharing, the agents expect to receive contingent payments from the system. Each agent is assumed to get one single contingent payment. The term “contingent payment” is general and can have various interpretations in different circumstances. For instance, it can refer to the risk exposure of an insurance company after risk sharing in the case of a reinsurance contract, or the investment risk in the case of a collective pension fund. Alongside there is also a long-lived *buffer* which makes the intertemporal money transfer possible.

The system starts at time t_0. Assume that altogether there are N contingent payments happening at time $t_1 \le t_2 \le \cdots \le t_N$, where N is some positive integer. C_n will stand for the contingent payment paid out from the system at time t_n. Let F_n be the buffer size at time t_n. X_n denotes the financial risk coming into the system from the agents from time t_{n-1} to t_n, that is, it is the sum of all the stochastic cash inflows from the agents from time t_{n-1} to t_n. The risk stream $X = (X_1, \dots, X_N)$ is defined in a financial market in which prices are given exogenously. The buffer is invested in a risky asset R which produces stochastic per-dollar gross return R_n from time t_{n-1} to t_n. Here the C_n's and F_n's are decision variables, and the X_n's and R_n's are the risks to be shared.

The X_n's and R_n's are random variables defined on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is the objective measure. $\mathbb{F}$ is the filtration generated by the X's and R's:

$$\mathbb{F} = \{\mathcal{F}_n | n = 1, \dots, N\}, \quad \mathcal{F}_n = \sigma\{(X_1, R_1), \dots, (X_n, R_n)\}.$$

There is also a risk-neutral measure $\mathbb{Q}$ defined on the probability space besides the objective measure $\mathbb{P}$. There is no need to assume the completeness of the market; any given risk-neutral measure $\mathbb{Q}$ will suffice. The only assumption is that the agents have agreed to adopt some probability measures $\mathbb{P}$ and $\mathbb{Q}$, or the measures are simply specified in a situation where a social planner is present. Let

$$\mathbb{E}_n[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_n].$$

It is assumed that the process $\{(X_n, R_n)\}$ is sequentially independent under $\mathbb{P}$ and $\mathbb{Q}$, that is, (X_t, R_t) and (X_s, R_s) are independent for $t \neq s$ under $\mathbb{P}$ and $\mathbb{Q}$. For $n = 1, \dots, N$, the random variables X_n and R_n need not be independent, and their joint distribution is known. As we are working on a finite probability space, the total number of outcomes of (X_n, R_n) is finite for all n. Illustrated by Fig. 2, the risks can be seen as a multinomial tree and every pair (X_n, R_n) can then be totally characterized by

$$\left\{ \left((X_n^{j_n}, R_n^{j_n}), \mathbb{P}(j_n), \mathbb{Q}(j_n) \right) \middle| j_n = 1, \dots, m_n \right\}$$

where $(X_n^{j_n}, R_n^{j_n})$ represents all the possible and distinct values of (X_n, R_n) and $\mathbb{P}(j_n), \mathbb{Q}(j_n)$ are the corresponding $\mathbb{P}$- and $\mathbb{Q}$-probabilities. A technical requirement is that for any $n = 1, \dots, N$

$$\mathbb{Q}\left(\{\omega \in \Omega | X_n(\omega) = \max X_n, R_n(\omega) = \max R_n\}\right) > 0, \tag{2.1}$$

which means that X_n and R_n can attain their maximum under $\mathbb{Q}$ simultaneously. Furthermore we assume that $R_n > 0$ for all n as the R's have the interpretation as the gross return of the asset R.

Write $J_n = j_1 j_2 \cdots j_n$ as the trajectory $\left((X_1^{j_1}, R_1^{j_1}), \dots, (X_n^{j_n}, R_n^{j_n})\right)$. Let $\mathcal{J}_n$ be the set of all the possible trajectories of (X, R) up to time t_n. $J_n j_{n+1}$ will denote any trajectory whose up-to-time-t_n part is J_n. In such a situation we write $j_{n+1} \in \mathcal{J}_n^{n+1}$ where $\mathcal{J}_n^{n+1}$ denotes the set of all the possible cases of (X_{n+1}, R_{n+1}).

The risk-neutral measure $\mathbb{Q}$ is used to price the risks X as well as the investment returns R. In this generic setting, write

$$x_n := \mathbb{E}^{\mathbb{Q}} X_n, \qquad 1 + r_n := \mathbb{E}^{\mathbb{Q}} R_n, \quad n = 1, \dots, N.$$

The x_n's are the market prices of the risks X and the r_n's are the risk-free returns implied by the pricing measure $\mathbb{Q}$. Please note that now and later we directly work with future values for convenience.

Note that the time points $\{t_0, t_1, \dots, t_N\}$ need not be equidistant. As shown in Fig. 1, two or more time points can be equal if there are more than one contingent payment paid out at the same time. In that case, say $t_{n-1} = t_n$ for some n, we shall have $X_n \equiv 0$ and $R_n \equiv 1$, because there will be no risks coming in and the buffer will not evolve with respect to asset return.