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The valuation of life contingencies: A symmetrical triangular fuzzy approximation[☆]

Jorge de Andrés-Sánchez[a,*], Laura González-Vila Puchades[b]

[a] Social and Business Research Laboratory, Rovira i Virgili University, Av. de la Universitat 1, 43204 Reus, Spain

[b] Department of Mathematics for Economics, Finance and Actuarial Science. Faculty of Economics and Business. University of Barcelona, Av. Diagonal 690, 08034 Barcelona, Spain

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ABSTRACT

This paper extends the framework for the valuation of life insurance policies and annuities by Andrés-Sánchez and González-Vila (2012, 2014) in two ways. First we allow various uncertain magnitudes to be estimated by means of fuzzy numbers. This applies not only to interest rates but also to the amounts to be paid out by the insurance company. Second, the use of symmetrical triangular fuzzy numbers allows us to obtain expressions for the pricing of life contingencies and their variability that are closely linked to standard financial and actuarial mathematics. Moreover, they are relatively straightforward to compute and understand from a standard actuarial point of view.

1. Introduction

Stochastic techniques are, beyond doubt, at the core of actuarial mathematics. However, in insurance decision-making problems, as well as in other areas related to economics and finance, much of the information is imprecise and vague, or relies heavily on subjective judgments and, so, it is not clearly measurable. For such information, the use of fuzzy set theory (FST) can represent a suitable alternative and/or a supplementary way to that of pure statistical methods as has been shown in De Wit (1982), Lemaire (1990), Ostaszewski (1993), Cummins and Derrig (1997), Andrés-Sánchez and Terceño (2003) and Shapiro (2004).

In the field of the financial pricing of insurance, FST has been used to model discount rates. Cummins and Derrig (1997), Derrig and Ostaszewski (1997) and Andrés-Sánchez (2014) do so in a non-life context, while Lemaire (1990), Ostaszewski (1993) and Betzuen et al. (1997) model discount rates for life insurance contingencies valuation. In these papers probabilities, however, are reduced to predefined frequencies and so, the financial pricing of insurance contracts is solved by applying the fuzzy financial mathematics developed by Buckley (1987). Anyway, when applying these methods probabilistic information is lost because random magnitudes are reduced to their mathematical expectation.

Shapiro (2009) exposes the concept of fuzzy random variables (FRVs) with Actuarial Science in view. Similarly, Huang et al. (2009) develop an individual risk model in which the cost of accidents is estimated using fuzzy numbers (FNs), while the number of claims follows a Poisson process. In the field of life insurance, Andrés-Sánchez and González-Vila Puchades (2012, 2014) develop a methodology in which discount rates are fuzzy whereas the mortality is strictly random. In these papers the stochastic modeling of life contingencies with deterministic discount rates and monetary amounts (see Wolthuis and Van Hoek, 1986 for a complete description) is extended to cases in which the discount rates are fuzzy and, so, the outcomes (the present value of insured life contingencies) are fuzzy sets. All these developments also rely

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* Corresponding author.
E-mail addresses: jorge.deandres@urv.cat (J. de Andrés-Sánchez), lgonzalezv@ub.edu (L. González-Vila Puchades).

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on the concept of FRVs. In their works, the authors do not assume any particular shape for fuzzy interest rates and consequently no closed expressions for those present values are developed. Likewise, only crisp unitary amounts are considered.

This paper extends the previous findings of Andrés-Sánchez and González-Vila Puchades (2012, 2014) in two ways. First, we also allow the insured amounts to be paid out by the insurance company to be uncertain and to be quantified with FNs, which is a more general framework. Note, that, in fact, these amounts may be linked to economic indexes, such as the consumer price index, a wage growth rate, etc. Likewise, if we are evaluating the underwriter's overall outcome of a policy, future maintenance costs, general and settlement expenses, etc. may not be known exactly *a priori*.

Second, we suppose that the amounts to be paid and the interest rates are fitted with symmetrical triangular fuzzy numbers (STFNs). Indeed, the use of these FNs is very common in the fuzzy literature. In a strictly actuarial context, we find Andrés-Sánchez and Terceño (2003), Shapiro (2013) and Heberle and Thomas (2014). Despite that under these hypothesis the present value of the analyzed life contingencies structures does not turn into a STFN, we will propose a STFN approximation that is relatively easy to implement and understand with standard actuarial skills since it relies on conventional statistical and financial concepts. In our opinion, the issue of maintaining the symmetrical triangular shape in the present value of insurance contract is relevant. Following Grzegorzewski and Pasternak-Winiarska (2014) complicated forms of FNs may cause unpleasant drawbacks in processing imprecise information modeled by these fuzzy structures including problems with calculations, computer implementation and in interpretation of the results. This is the reason why a suitable approximation of FNs is an interesting alternative to substitute the original "input" membership functions by another "outputs" which are simpler or more regular and hence more convenient for further tasks. In this sense Grzegorzewski and Mrówka (2005) indicate that triangular approximation can be considered a more complete kind of defuzzification than simply reducing a FN to a crisp representative value given that performing defuzzification early may result in a loss of too much information and it is better to process fuzzy information as long as possible. Bearing in mind this idea, we are looking for simplification in the computational process and the interpretation of the results on the one hand but, on the other, we do not want to simplify too much the original information.

We structure the rest of our paper as follows. In Section 2 we describe the concepts and instruments of FST used in our developments: FNs and FRVs. We then develop a STFN approximation for the present value derived from STFN cash-flows and discount rates with a straightforward financial interpretation. In Sections 4 and 5 we introduce the use of FRV with STFN outcomes to price life contingencies. We conclude our paper with a summary of the main conclusions and possible extensions.

2. Fuzzy numbers and fuzzy random variables

2.1. Fuzzy numbers and fuzzy arithmetic

In this section we describe the basic concepts of FST and FNs and so present the basic notation used throughout this paper. The basic concept on which FST is based is *fuzzy set*. A fuzzy set $\tilde{A}$ can be defined as $\tilde{A} = \left\{\left(x, \mu_{\tilde{A}}(x)\right) | x \in X\right\}$, where $\mu_{\tilde{A}}$ is known as the membership function and is a mapping from the referential set X to the interval $[0, 1]$, i.e. $\mu_{\tilde{A}} : X \to [0, 1]$. Therefore, 0 indicates non-membership in the fuzzy set $\tilde{A}$ and 1 indicates absolute membership. Alternatively, a fuzzy set can be represented by its α-level sets or α-cuts. An α-cut is a crisp set A_α, where $A_\alpha = \left\{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\right\}, \forall \alpha \in (0, 1]$, with the convention that $A_{\alpha=0}$ is the closure of the support of $\tilde{A}$, i.e. all $x \in X$ that $\mu_{\tilde{A}}(x) \geq 0$.

A fuzzy number is a fuzzy set $\tilde{A}$ defined over the set of real numbers and it is a fundamental concept of FST for representing uncertain quantities. It is normal, i.e. $\max_{x\in X} \mu_{\tilde{A}}(x) = 1$, and convex, that is, its α-cuts are closed and bounded intervals. So, they are $A_\alpha = \left[\underline{A}(\alpha), \overline{A}(\alpha)\right]$, where $\underline{A}(\alpha)$ ($\overline{A}(\alpha)$) are continuously increasing (decreasing) functions of the membership level $\alpha \in [0, 1]$. A FN can be interpreted as a fuzzy quantity approximately equal to the real number for which the membership function takes the value 1. In this paper we use *symmetrical triangular fuzzy numbers*, which we denote as $\tilde{A} = (A, r_A)$. The value A is the core (mode or center) and it can be understood as the most reliable value of the FN, i.e. $\mu_{\tilde{A}}(A) = 1$. Likewise $r_A \geq 0$ is the spread or radius and indicates the variability of $\tilde{A}$ respect its mode A. Thus, the membership function and its corresponding α-cuts are:

$$\mu_{\tilde{A}}(x) = \max\left\{0, 1 - \frac{|x - A|}{r_A}\right\}$$

$$A_\alpha = \left[\underline{A}(\alpha), \overline{A}(\alpha)\right] = \left[A - r_A(1-\alpha), A + r_A(1-\alpha)\right]. \tag{1}$$

The *expected value* of the FN $\tilde{A}$, $EV\left(\tilde{A}; \lambda\right)$, is a representative real value of this FN that was developed in Campos and González (1989). This concept allows us to introduce the risk aversion of the decision maker with a coefficient $0 \leq \lambda \leq 1$ in such a way that:

$$EV\left(\tilde{A}; \lambda\right) = (1-\lambda)\int_0^1 \underline{A}(\alpha)\, d\alpha + \lambda \int_0^1 \overline{A}(\alpha)\, d\alpha. \tag{2a}$$

In (2a), λ graduates the importance of the lower and upper extremes of A_α when defuzzifying $\tilde{A}$. So, the greater the risk aversion of the decision maker is, the greater λ is. For example, in a non-life claim reserving context Heberle and Thomas (2014) and Andrés-Sánchez (2014) use this criteria to defuzzify the value of reserves previously given by FNs in such a way that $\lambda > 0.5$ for a risk-averse criteria for reserving.

So, for a STFN $\tilde{A} = (A, r_A)$ it is straightforward to check that:

$$EV\left(\tilde{A}; \lambda\right) = A + r_A\left(\lambda - \frac{1}{2}\right). \tag{2b}$$

Let $f(\cdot)$ be a continuous real valued function of n-real variables x_j, $j = 1, 2, \ldots, n$, and let $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ n FNs. Then Zadeh's *extension principle* in Zadeh (1965) allows us to define a FN $\tilde{B}$ induced by the FNs $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ through $f(\cdot)$ as $\tilde{B} = f\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right)$.

Although it is usually impossible to obtain the membership function of $\tilde{B}$, it is often possible to obtain a closed expression for its α-cuts, B_α. If $f(\cdot)$ is increasing with respect to the first m variables, $m \leq n$, and decreases in the last $n - m$ variables, Buckley and Qu (1990) demonstrate that:

$$\begin{aligned} B_\alpha &= \left[\underline{B}(\alpha), \overline{B}(\alpha)\right] \\ &= \left[f\left(\underline{A}_1(\alpha), \underline{A}_2(\alpha), \ldots, \underline{A}_m(\alpha), \overline{A}_{m+1}(\alpha),\right.\right. \\ &\qquad \left.\overline{A}_{m+2}(\alpha), \ldots, \overline{A}_n(\alpha)\right), \\ &\qquad f\left(\overline{A}_1(\alpha), \overline{A}_2(\alpha), \ldots, \overline{A}_m(\alpha), \underline{A}_{m+1}(\alpha),\right. \\ &\qquad \left.\left.\underline{A}_{m+2}(\alpha), \ldots, \underline{A}_n(\alpha)\right)\right]. \end{aligned} \tag{3}$$

When $f(\cdot)$ is simply a linear combination of its variables $\sum_{j=1}^{n} k_j x_j$, $k_j \in \Re$, $j = 1, 2, \ldots, n$, the result of evaluating this function with $\tilde{A}_j = \left(A_j, r_{A_j}\right)$, $j = 1, 2, \ldots, n$, is a STFN $\tilde{B} = (B, r_B)$, where:

$$B = \sum_{j=1}^{n} k_j A_j, \qquad r_B = \sum_{j=1}^{n} \left|k_j\right| r_{A_j}.$$