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Optimal multivariate quota-share reinsurance: A nonparametric mean-CVaR framework

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ABSTRACT

In this paper, the Conditional Value-at-Risk (CVaR) is adopted to measure the total loss of multiple lines of insurance business and two nonparametric estimation methods are introduced to explore the optimal multivariate quota-share reinsurance under a mean-CVaR framework. While almost all the existing literature on optimal reinsurance are based on a probabilistic derivation, the present paper relies on a statistical analysis. The proposed optimal reinsurance models are directly formulated on empirical data and no explicit distributional assumption on the underlying risk vector is required. The resulting nonparametric reinsurance models are convex and computationally amenable, circumventing the difficulty of computing CVaR of the sum of a generally dependent random vector. Statistical consistency of the resulting estimators for the best CVaR is established for both nonparametric models, allowing empirical data to be generated from any stationary process satisfying strong mixing conditions. Finally, numerical experiments are presented to show that a routine bootstrap procedure can capture the distributions of the resulting risk measures well for independent data.

1. Introduction

Reinsurance can be an effective way of managing risk by transferring risk from an insurer (also referred to as the cedent) to a second insurance carrier (referred to as the reinsurer). The study of optimal reinsurance design has attracted great attention from both academicians an practitioners since the seminar work of Borch (1960). In the past half-century, a variety of optimal reinsurance designs have been devised by either minimizing certain risk measure of an insurer's risk exposure or maximizing the expected utility of the final wealth of a risk-averse insurer; see, for example, Borch (1960), Arrow (1963), Raviv (1979), Huberman et al. (1983), Young (1999), Kaluszka (2001) and Kaluszka and Okolewski (2008), and references therein. More recently, the Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) based optimal reinsurance designs have been extensively studied due to the prevalent use of the two risk measures in financial and insurance practice; see, for instance, Gajek and Zagrodny (2004), Huang (2006), Cai et al. (2008), Balbás et al. (2009), Cheung (2010), Tan et al. (2011), Asimit et al. (2013), Chi and Weng (2013), Cheung et al. (2014a) and Tan and Weng (2014), just to name a few.

Almost all the optimal reinsurance models in the aforementioned literature are for a single random variable, and the results are therefore applicable only to aggregate loss. In the insurance practice, however, an insurer with multiple lines of business often purchases reinsurance separately for each line of business. Therefore, from the perspective of enterprise risk management, it is prudent for the insurer to investigate the strategies to buy reinsurance on each individual line of business and attain the optimality in certain integrated sense. Due to the inherent dependence among the risks of individual lines, optimal multivariate reinsurance problems are usually difficult to be solved. Under certain special dependence structure, Denuit and Vermandele (1998) and Cai and Wei (2012) show that an excess-of-loss reinsurance is optimal for the expected value reinsurance premium principle. Cheung et al. (2014b) propose a minimax model and its solution is also in favor of excess-of-loss reinsurance. Zhu et al. (2014) study the multiple optimal reinsurance by minimizing the multivariate lower-orthant Value-at-Risk, and shows that a two-layer reinsurance for each line is optimal for various reinsurance premium principles.

In the present paper, we consider the insurer's optimal decision on purchasing multivariate quota-share reinsurance against its losses from multiple lines of business, in view that quota-share

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reinsurance is one of the main proportional reinsurance contracts and it remains among the most popular forms of reinsurance in insurance practice, particularly for life insurance. Under a quota-share reinsurance treaty, the reinsurer takes a stated percentage share of each policy that an insurer underwrites. The reinsurer receives that stated percentage of the premiums and pays the stated percentage of claims. Therefore, the determination of the optimal multiple quota-share reinsurance boils down to deciding on the optimal percentage share (called quota-share coefficient) for each line of business. In the present paper, the optimal reinsurance treaties are studied under a mean-CVaR framework, where the Conditional Value-at-Risk (CVaR) is adopted to measure the total loss of multiple lines of insurance business and a constraint on the insurer's expected net profit is imposed.

The proposed mean-CVaR optimal reinsurance model is theoretically appealing as it allows the insurer to balance its risk and reward in exploring optimal reinsurance purchase strategies. Nevertheless, such theoretical model is subject to two major issues. First, its formulation heavily relies on the distribution of the underlying risk vector, which in practice is uncertain and needs to be estimated from empirical data. Second, even though the distribution of the underlying risk vector is explicitly known, the analysis of optimal reinsurance is often intimidatingly challenging because it involves computing the CVaR of the aggregate risk from a generally dependent random vector. In the present paper, we follow the idea of Weng (2009) (also see, Tan and Weng, 2014) and formulate optimal reinsurance models directly from empirical data, leading to fully nonparametric models. We propose two types of nonparametric models upon the theoretical mean-CVaR framework and a representation result of CVaR from Rockafellar and Uryasev (2002). The first one is directly formulated under the empirical measure and it leads to a linear programming problem, which we refer to as "linear programming model" throughout the paper. Second, we propose a kernel estimation for the risk measure CVaR and the resulting nonparametric model becomes a convex programming. Such a model is termed as "kernel-based model" throughout the paper. The actuarial literature seem quite sparse on the non-parametric estimation of risk measures. To the best knowledge of the authors, Peng et al. (2015) and Wang and Peng (2016) are two of such recent papers.

Generally, there are four solution methods for a stochastic optimization problem (such us our mean-CVaR optimal reinsurance): sample average approximation (SAA), stochastic approximation, response surfaces, and metamodels. The non-parametric methods proposed for the mean-CVaR optimal reinsurance model in this paper can be categorized into the group of SAA. The idea of SAA is that the objective function is approximated by the average of the realizations, and then the problem becomes deterministic so that many deterministic search methods can be used to solve the approximate problems; see Robinson (1996), Shapiro and Wardi (1996) and Shapiro et al. (2002) and references therein.

One integral aspect to address an optimization problem using the SAA is the convergence of the approximation to the true optimal value in a stochastic optimization problem. In this paper, the best risk measure level solved from both nonparametric models is formally shown convergent to the theoretically best level for empirical data generated from any stationary process satisfying certain moment and strong mixing conditions. These convergence results cannot be derived trivially from the existing literature on SAA for stochastic optimization. Firstly, the realizations or observations are always assumed independent and identically distributed (i.i.d.) in the literature and most theoretical results such as consistency and convergence rate follow directly from the standard law of large numbers and central limit theorem due to the simple assumptions; for example, see Shapiro (1991), Rubinstein and Melamed (1998) and Shapiro et al. (2009). In contrast, our convergence results apply to empirical data which are generated from any stationary process satisfying certain moment and strong mixing conditions (see Assumption 6.1 in the sequel), which includes the i.i.d. as a special case and better reflects the dependence nature of insurance data.

Secondly, in order to establish convergence results, most existing literature on SAA exploit the Lagrangian duality method so that the randomness is only present in the objective function, and eventually conduct asymptotic analysis via functional spaces; see, for example, Shapiro (1991). It is well known that such a functional space based methodology is typically challenging, if not impossible, to be applied to a general dependent case like the strong mixing dependence in our models. So, we take a completely different way to prove the convergence results. We do not apply the Lagrangian duality method in our proof; instead, we keep the randomness in both objective and feasible set. To deal with the random feasible set, we follow the idea of perturbation analysis, where bounds of the random feasible set are applied and they converge to the true deterministic feasible set along with the increase of sample size. Such an idea can be seen in Lemma 6.3 and the proof of Proposition 6.1.

To end the section, we summarize the advantages of our proposed empirically data-based models. First, they are completely data-based and no explicit assumption is required on the distribution of the loss random vector. Second, the proposed nonparametric models bypass the technical obstacle of computing the CVaR of the aggregate loss from a generally dependent random vector, which cannot be circumvented in a theoretical model. Third, the proposed nonparametric models are computationally amenable and can be solved efficiently as either a linear programming or a convex programming. Fourth, statistical consistency results are established to provide theoretical support to our proposed nonparametric models. Our numerical simulation with exponential and Pareto marginal distributions and Gaussian copula illustrate that both the linear programming model and kernel-based model work well for reasonably large sample size, with the latter performing better than the former. Finally, our numerical experiments in Section 7.4 demonstrate that a typical bootstrap procedure can capture the distributions of the resulting risk measures well for independent sample.

The rest of the paper proceeds as follows. The mathematical formulation of multiple quota-share reinsurance is introduced in Section 2, and the theoretical mean-CVaR reinsurance model is defined in Section 3. The linear programming model and the kernel-based model are specified in Section 4. Consistency properties for the proposed nonparametric reinsurance models are shown in Section 6. Numerical studies are presented in Section 7 for performance comparison between the linear programming model and kernel based model. In the same section, a bootstrap procedure is proposed for uncertainty quantification of the resulting risk measures (which are random variables) from the two non-parametric models. Section 8 concludes the paper. Finally, some technical lemmas are given in Appendix A, and some proofs are collected in Appendices B and C.

2. Multiple quota-share reinsurance

Let $X = (X_1, X_2, \dots, X_p)$ be the claim (aggregate) loss vector on p lines of an insurer's business, where X_i denotes the aggregate nonnegative loss random variable (in the absence of reinsurance) the insurer is subject to in its ith line of business with cumulative distribution function $F_i(x) = \mathbb{P}(X_i \le x)$, survival function $\overline{F}_i(x) = 1 - F_i(x) = \mathbb{P}(X_i > x)$ and mean $\mu_i = \mathbb{E}[X_i] < \infty, i = 1, \dots, p$.

The insurance premium is often calculated by the expected value principle in insurance practice, because the safety loading on the top of the net premium can be justified by estimating the