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On optimal dividends with exponential and linear penalty payments

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ABSTRACT

We study the optimal dividend problem where the surplus process of an insurance company is modelled by a diffusion process. The insurer is not ruined when the surplus becomes negative, but penalty payments occur, depending on the level of the surplus. The penalty payments shall avoid that losses can rise above any number and can be seen as a preference measure or costs for negative capital. As examples, exponential and linear penalty payments are considered. It turns out that a barrier dividend strategy is optimal.

1. Introduction

In a classical approach the risk of an insurance company is measured by the probability of ruin. For optimal decisions, the probability of ruin is minimised – for example by reinsurance and/or investments – in order to increase the solvency of an insurance company. This problem was considered for example in Azcue and Muler (2014), Hipp and Plum (2000), Schmidli (2002) and Schmidli (2008), where further references can be found. The ruin probability approach neglects the time value of money and supposes that the surplus tends to infinity. A further approach distributes dividends to the shareholders, where the goal is to maximise the expected discounted dividends until ruin. The formulation of the dividend problem in a discrete time framework goes back to de Finetti (1957). After that, Gerber (1969) considered the problem for the Cramér–Lundberg model. In a more recent paper, Gerber and Shiu (2004) analysed the dividend approach in a diffusion model. Avanzi (2009) gave an overview on the actuarial research that followed de Finetti's original paper. However, the disadvantage of the dividend approach is that, under the optimal strategy, ruin occurs almost surely. Therefore, the idea of capital injections rises. Whenever the surplus becomes negative, the shareholders have to inject capital in order to avoid ruin. Originally proposed by Dickson and Waters (2004) for barrier dividend strategies, Kulenko and Schmidli (2008) considered optimal dividends and capital injections policies.

All of the approaches above have one thing in common: whenever the surplus becomes negative, the insurer either has to inject capital or ruin occurs. During the financial crisis of 2007–2008, it was observed, that some companies continue with the business, although they had large losses for a long period. Often, the regulator intervenes in order to avoid that a company goes out of business. Besides several banks, for example Bradford and Bingley, Dexia, Lehman Brothers and Hypo Real Estate, the insurance company AIG was also concerned of the financial crisis and the solvency was only ensured by interventions of the regulators. Therefore, it is more realistic to allow negative surplus. Nevertheless, neither theoretical nor practical it makes sense if the losses of the insurer can rise above any number. In order to avoid this we introduce penalty payments. There are two interpretations of these penalty payments:

(1) Introducing a preference measure
The value function may be seen as a technical tool to investigate the profitability or risk of a portfolio. Further, with the value function the effect of possible interventions by the risk manager can be measured. In this sense, the dividends measure the profitability, the penalty is a preference measure where large capital is preferred to lower (or negative) capital. In the context of negative surplus, Albrecher et al. (2011c) also

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introduced a rather technical bankruptcy function, where the probability of bankruptcy is a function of the level of negative surplus. This was further discussed in Albrecher et al. (2011a,b, 2013) and Albrecher and Lautscham (2013). It turns out that it is hard to obtain explicit solutions in the framework of Albrecher et al. (2011c). A similar discussion can also be found in Schmidli (2016, forthcoming).

(2) Penalty interests for negative capital

The penalty payment can be seen as real costs. For example, if the surplus of an insurance line becomes negative, capital has to be borrowed from other lines of insurance or from other companies of a holding. Then this capital cannot be invested anymore leading to a loss of investment return. This is in particular understandable for a linear penalty function. Linearity, however, is not necessary because lower negative capital might cause higher costs. Interest payments for negative surplus was also considered by Gerber (1971), Embrechts and Schmidli (1994) and Schmidli (1994). Note, that in the latter papers costs for interest are directly charged to the surplus process of the insurance line.

Similar models were also treated by Højgaard and Taksar (1997) as well as Marciniak and Palmowski (2015). In Højgaard and Taksar (1997) a discounted reward for surplus was optimised until the time of ruin. Marciniak and Palmowski (2015) considered an optimal dividend distribution problem for an insurance company with surplus-dependent premium.

For a surplus level of x, we model the penalty payments to apply at rate $\phi(x)$. If the economic situation of the insurer deteriorates, the penalty payments increase. Moreover, the penalty payments are always positive and vanish as the surplus tends to infinity. Thus, ϕ should be a decreasing and positive function with $\phi(x) \to 0, x \to \infty$. In addition, dividends may be paid. The value of the surplus process $\{X_t\}$ with accumulated dividends $\{D_t\}$ is then

$$\mathbb{E}\Big[\int_0^\infty e^{-\delta t}\, dD_t - \int_0^\infty e^{-\delta t}\phi(X_t^D)\, dt\Big],$$

where $\delta > 0$ is a preference parameter. That is, dividends today are preferred to dividends tomorrow, and cost tomorrow are preferred to costs today. Our goal will be to maximise this value by choosing an optimal dividend policy.

In our model the size of the portfolio and the economic environment is fixed for all the future. In reality, of course, the insurer would like to underwrite new risk. Further, the economic environment and with it the claims size distribution and the frequency of claims changes. Dependent on the market, also the insurer would adjust the premiums. But we have to see the value function as a technical tool helping the manager to take decisions.

This paper is organised as follows. In the second section we introduce the mathematical framework and we motivate the Hamilton–Jacobi–Bellman (HJB) equation. In Section 3 we prove the verification theorem. Section 4 considers the dividend problem with an exponential penalty function $\phi(x) = \alpha\exp(-\beta x)$, where $\alpha, \beta > 0$. Here, the value function exists only if $r_2 < -\beta$, where r_2 is the negative solution to the equation $\sigma^2 r^2 + 2\mu r - 2\delta = 0$. If $r_2 \ge -\beta$ no optimal strategy does exist. For $r_2 < -\beta$, we show that the optimal strategy is a barrier strategy and determine the optimal barrier. Section 5 studies a linear penalty function $\phi(x) = -\alpha x$ for some $\alpha > 0$ if $x < 0$ and $\phi(x) = 0$ if $x \ge 0$. An optimal strategy does only exist if $\delta < \alpha$, where δ denotes the discounting factor. In this case the optimal strategy is a barrier strategy and the optimal barrier is given by $b^* = 1/r_2 \log(\delta/\alpha)$. If $\delta \ge \alpha$, the preference parameter is larger than the slope of the penalty payments and it is optimal to pay an infinite amount of dividends.

2. The model and the HJB equation

We assume that the surplus of the insurance company follows a diffusion approximation

$$X_t = x + \mu t + \sigma W_t, \quad t \ge 0, \tag{2.1}$$

where $x \in \mathbb{R}$ denotes the initial capital, W_t a standard Brownian motion and $\mu, \sigma > 0$. This process is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The information is given by the natural filtration $\mathcal{F}_t$ of the Brownian motion. Let D_t be adapted and denote the accumulated dividend payments until time t. Then, the controlled surplus process is given by

$$X_t^D = X_t - D_t.$$

We allow all increasing càdlàg processes D with $D_{0-} = 0$. The value of a strategy D is defined by

$$V^D(x) = \mathbb{E}\Big[\int_0^\infty e^{-\delta t}\, dD_t - \int_0^\infty e^{-\delta t}\phi(X_t^D)\, dt \,\Big|\, X_0^D = x\Big]. \tag{2.2}$$

The decreasing function ϕ is the penalty function fulfilling $\phi(x) \to 0$ as $x \to \infty$. We further assume that ϕ is convex. The set of adapted strategies is denoted by $\mathcal{D}$ and the (optimal) value function is defined by

$$V(x) = \sup_{D\in\mathcal{D}} V^D(x).$$

We aim to find a strategy D^* such that

$$V^{D^*}(x) = V(x).$$

The costs are bounded by the costs obtained if no dividends are paid. We therefore have to assume

$$\int_0^\infty e^{-\delta t}\mathbb{E}[\phi(X_t)]\, dt < \infty.$$

Otherwise, the value function would be minus infinity. Moreover, we assume that

$$\phi(x) - \phi(y) > \delta(y - x) \tag{2.3}$$

for $x < y < x_0$ and some $x_0 \in \mathbb{R}$ in order that it is not optimal to pay an infinite amount of dividends. Since ϕ is assumed to be convex, this means that there is an $x \in \mathbb{R}$ such that $\phi'(x) < -\delta$.

It is well-known that the optimal dividend strategy in the model without penalty payments is a barrier strategy. A barrier strategy D is characterised by a barrier b, where all the surplus above b is paid as dividends and whenever the surplus is below b, no dividends are paid. This means that

$$D_t = \max\Big(\sup_{0\le s\le t} X_s - b, 0\Big).$$

Fig. 1 shows a sample path of a surplus process controlled by a barrier strategy. We expect that in our problem the optimal strategy is also a barrier strategy. Then, $V(x) = V(b) + x - b$ for $x \ge b$. If $x < b$ let $\tau^b = \inf\{t > 0 : X_t > b\}$ and $h > 0$. We find

$$V(x) = \mathbb{E}\Big[e^{-\delta(\tau^b\wedge h)}V(X^{D^*}_{\tau^b\wedge h}) - \int_0^{\tau^b\wedge h} e^{-\delta t}\phi(X_t^{D^*})\, dt\Big].$$

Assuming that V is twice continuously differentiable, one easily can show by Itô's formula that

$$\frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \delta V(x) - \phi(x) = 0, \tag{2.4}$$

if $x < b$. This motivates the HJB equation

$$\max\left(\frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \delta V(x) - \phi(x), 1 - V'(x)\right) = 0. \tag{2.5}$$