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# Asymptotic analysis for target asset portfolio allocation with small transaction costs

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ABSTRACT

In this paper we discuss the asset allocation in the presence of small proportional transaction costs. The objective is to keep the asset portfolio close to a target portfolio and at the same time to reduce the trading cost in doing so. We derive the variational inequality and prove a verification theorem. Furthermore, we apply the second order asymptotic expansion method to characterize explicitly the optimal no transaction region when the transaction cost is small and show that the boundary points are asymmetric in relation to the target portfolio position, in contrast to the symmetric relation when only the first order asymptotic expansion method is used, and the leading order is a constant proportion of the cubic root of the small transaction cost. In addition, we use the asymptotic results for the boundary points and obtain an expansion for the value function. The results are illustrated in the numerical example.


## 1. Introduction

In pension and insurance fund management it is often necessary to allocate the fund in different asset classes with fixed proportions of wealth invested in each one of them. This may be due to the regulatory requirement, asset diversification, liability structure, etc., see Meyer and Meyer (2005) and Dionne (2013). Such a strategy is called constant mix (or rebalancing of investments) trading strategy, see Ang (2014). If the market is complete, it is easy to achieve the fixed proportion of wealth in a specific asset by continuously trading the underlying asset, but it is impractical in the presence of transaction costs (brokerage fees, taxes, etc.). The fund manager then faces two conflicting objectives: reducing the total transaction cost and reducing the tracking error (dispersion from the target), see Grinold and Kahn (2000). This paper discusses optimal trading strategies in the presence of small transaction costs. The problem is related to utility maximization with transaction costs. We next give a literature review on the subject.

The work of Merton (1969) is the starting point of continuous-time utility based portfolio theory. With the help of the stochastic control theory, the portfolio problem can be formulated as a Hamilton–Jacobi–Bellman (HJB) equation, which can be solved explicitly for a hyperbolic absolute risk aversion investor. The corresponding optimal investment strategy involves continuously rebalancing the portfolio to maintain a constant fraction of total wealth in each asset during the whole investment period. However, this optimal policy is unrealistic in the presence of transaction costs. Magill and Constantinides (1976) are the first to incorporate proportional transaction costs into Merton's model. Their heuristic analysis for the infinite horizon investment and consumption problem gives a fundamental insight into the optimal strategy and the existence of the no transaction region. Davis and Norman (1990) provide a rigorous mathematical analysis for the same problem by applying the stochastic control theory. Using "continuous control" (consumption) and "singular control" (transaction), they show that the investor's optimal trading strategy is to maintain the portfolio position inside the no transaction region. If the initial portfolio position is outside the no transaction region, the investor should immediately sell or buy stock in order to move to its boundary. The investor then trades only when the portfolio position is at the boundary of the no transaction region, and only as much as necessary to keep it from exiting the no transaction region, while no trading occurs in the interior of the region. The optimal policies are determined by the solution of a free boundary problem, where the free boundaries correspond to the optimal buying and selling policies. Shreve and Soner (1994) generalize the results of Davis and Norman (1990) with the theory of viscosity solutions.

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In practice transaction costs are small relative to values of transactions. In the limit of small transaction costs, Atkinson and Wilmott (1995) apply the perturbation method to derive an approximate solution to a model with transaction cost of a fixed fraction of portfolio value in Morton and Pliska (1995). Janeček and Shreve (2004) provide a rigorous derivation of the asymptotic expansions of the value function and boundaries of the no transaction region for an investor with the power utility. Bichuch (2011) presents a rigorous proof for a finite horizon case. All aforementioned papers use the stochastic control methods. Some recent papers obtain the power series expansions of arbitrary order for the optimal value function and the boundaries of the no transaction region with the duality theory and the shadow price method, see Gerhold et al. (2012, 2014) for details and references therein.

Rogers (2004) observes that the impact of small transaction costs consists of two parts: the direct cost incurred by actual trading and the displacement cost due to deviating from the frictionless target position. Leland (2000) postulates a "cost function" as the discounted sum of the trading cost and the tracking error cost. Inspired by these works we formulate the target asset allocation problem with transaction costs as a cost minimization problem made of two parts, similar to those of Leland (2000). We prove a verification theorem for optimality of the local-time trading strategy. We use the Magnus expansion to characterize the solution of non-autonomous and non-homogeneous systems of ordinary differential equations (ODEs). We apply the first and second order asymptotic expansion method to describe explicitly the optimal no transaction region when the transaction cost is small, which is not discussed in Leland (2000). We show that the boundary points are asymmetric in relation to the target portfolio position, in contrast to the symmetric relations when only the first order asymptotic expansion method is used, and the leading order is a constant proportion (depending on the target asset portfolio) of the cubic root of the small transaction cost. The results and methods discussed in this paper can provide useful insights for insurance and pension fund managers in making asset allocation decisions in the presence of proportional transaction costs.

The paper is organized as follows. Section 2 describes the model and the cost minimization problem. Section 3 discusses the HJB variational inequality and the verification theorem and applies the Magnus expansion method to characterize the optimal solution in the no transaction region. Section 4 performs asymptotic analysis of the no transaction region and the value function with respect to the transaction cost parameter and shows that the boundary points are symmetric to the given target portfolio level with the first order asymptotic expansion method but are asymmetric with the second order asymptotic expansion method. Section 5 gives numerical examples of the main results. Section 6 concludes. Appendices A–D contains the proofs of the theorems.

## 2. Problem formulation

Assume $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ is a filtered probability space and the market consists of two securities: one riskless asset $S^0$ paying a fixed interest rate $r$, i.e., $S^0_t = e^{rt}$, and one risky asset $S$ following a geometric Brownian motion process

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion with $\{\mathscr{F}_t\}_{t\geq 0}$ being the natural filtration of $W$ and satisfying the usual conditions, $\mu$ $(\mu > r)$ and $\sigma^2$ are positive constants representing the instantaneous rate of return and variance of the stock, respectively. Assume the initial position of an agent is $x$ dollars in the money market and $y$ dollars in stock. Assume there is a proportional transaction cost in the sense that the investor pays a fixed fraction $\epsilon$ of the amount transacted on buying or selling the stock.

A *trading strategy* is any pair $(L_t, M_t)_{t\geq 0}$ of non-decreasing and right continuous adapted processes with $L_{0^-} = M_{0^-} = 0$. $L_t$ and $M_t$ represent the cumulative dollar values of buying and selling the stock respectively up to time $t$.

Denote by $X_t$ and $Y_t$ the monetary values of the riskless and risky positions, respectively. The self-financing condition and the dynamics of $S^0_t$ and $S_t$ imply that

$$dX_t = rX_t dt - (1+\epsilon)dL_t + (1-\epsilon)dM_t,$$
$$dY_t = \mu Y_t dt + \sigma Y_t dW_t + dL_t - dM_t$$

with $X_{0^-} = x$, $Y_{0^-} = y$, where $\epsilon \in (0, 1)$ is a constant proportional transaction cost per dollar trading. The dollar transaction cost at time $t$ is given by $\epsilon dL_t + \epsilon dM_t$. Note that

$$X_0 = x - (1+\epsilon)L_0 + (1-\epsilon)M_0,$$
$$Y_0 = y + L_0 - M_0$$

may differ from $X_{0^-}$, $Y_{0^-}$ because of the initial transaction at time 0.

Define the *solvency region*

$$\mathscr{S} := \{(x, y) \in \mathbb{R}^2 : x + (1+\epsilon)y \geq 0, x + (1-\epsilon)y \geq 0\}.$$

We can re-parameterize the problem by introducing new variables $w_t = X_t + Y_t$ (the total wealth at time $t$) and $\pi_t = Y_t/w_t$ (the fraction of total wealth held in stock at time $t$). The return of wealth, using the dynamics for $X_t$ and $Y_t$, can be calculated to follow

$$dw_t = [r + (\mu - r)\pi_t]w_t dt + \sigma \pi_t w_t dW_t - \epsilon dL_t - \epsilon dM_t$$

with $w_{0^-} = x + y$, denoted by $w$. When there is an initial transaction $w_0 = w - \epsilon(L_0 + M_0)$. The proportion of wealth in stock, $\pi_t$, satisfies the following stochastic differential equation

$$\begin{aligned} d\pi_t &= (\mu - r - \sigma^2\pi_t)\pi_t(1-\pi_t)dt \\ &\quad + \sigma\pi_t(1-\pi_t)dW_t + d\Pi^+_t - d\Pi^-_t, \end{aligned}$$

where $d\Pi^+_t = (1+\epsilon\pi_t)dL_t/w_{t^-}$, $d\Pi^-_t = (1-\epsilon\pi_t)dM_t/w_{t^-}$, and $\pi_{0^-} = y/w$, denoted by $\pi$. It can be easily checked that

$$(w_{t^-} - \epsilon dL_t)(\pi_{t^-} + d\Pi^+_t) - w_{t^-}\pi_{t^-} = dL_t$$
$$(w_{t^-} - \epsilon dM_t)(\pi_{t^-} - d\Pi^-_t) - w_{t^-}\pi_{t^-} = -dM_t,$$

which imply that $d\Pi^+_t$ and $d\Pi^-_t$ are the instantaneous absolute changes of $\pi_t$ at time $t$ as a result of buying and selling. Note that $\Pi^+_0$ and $\Pi^-_0$ may not be zero due to possible transactions at time 0.

The solvency region $\mathscr{S}$ can be transformed as an interval with respect to $\pi$:

$$\mathscr{S} = \{\pi \in \mathbb{R} : -1/\epsilon \leq \pi \leq 1/\epsilon\}.$$

A trading strategy $(\Pi^+, \Pi^-)$ is *admissible* if $(\Pi^+, \Pi^-)$ ensures $\pi_t \in \mathscr{S}$ for all $t \geq 0$ and satisfies

$$\mathbb{E}_\pi\left[\int_0^\infty e^{-\rho t} d\Pi^+_t\right] < \infty \quad \text{and} \quad \mathbb{E}_\pi\left[\int_0^\infty e^{-\rho t} d\Pi^-_t\right] < \infty, \quad (1)$$

where $\rho$ is a discount factor and $\mathbb{E}_\pi$ the conditional expectation operator with $\pi_{0^-} = \pi$. Condition (1) guarantees finite expected value of discounted total changes of risky proportions due to transactions. It rules out strategies with infinite discounted amount of transactions. The set of all admissible strategies given initial position $\pi$ is denoted by $\mathcal{A}(\pi)$. Note that $\rho$ is not necessarily equal to $r$, the riskfree interest rate, as $\rho$ is a subjective discount factor used by a portfolio manager for future transactions or opportunity costs, whereas $r$ is an objective one used in the market as a whole.

The trading strategy $(\Pi^+, \Pi^-)$ is a control on the state process $\pi$. It requires a portfolio manager to monitor the risky proportion process and make the trading decision based on the shift of the portfolio's risky position to the target position. In this paper we focus on the asset allocation problem in which the target asset ratio $\pi^*$ (for the risky asset) is given exogenously.