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Reducing model risk via positive and negative dependence assumptions

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ABSTRACT

We give analytical bounds on the Value-at-Risk and on convex risk measures for a portfolio of random variables with fixed marginal distributions under an additional positive dependence structure. We show that assuming positive dependence information in our model leads to reduced dependence uncertainty spreads compared to the case where only marginals information is known. In more detail, we show that in our model the assumption of a positive dependence structure improves the best-possible lower estimate of a risk measure, while leaving unchanged its worst-possible upper risk bounds. In a similar way, we derive for convex risk measures that the assumption of a negative dependence structure leads to improved upper bounds for the risk while it does not help to increase the lower risk bounds in an essential way. As a result we find that additional assumptions on the dependence structure may result in essentially improved risk bounds.

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1. Preliminaries and motivation

The problem of assessing the model risk associated with the risk measurement of a high dimensional portfolio has recently gathered a lot of interest in the actuarial and financial literature. To set a mathematical framework, we assume that a financial institution holds a d-dimensional risk portfolio over a fixed time period. This risk portfolio is represented by a random vector $\boldsymbol{X} = (X_1, \dots, X_d)$ on a standard atomless probability space $(\Omega, \mathcal{F}, P)$. The total loss exposure associated with $\boldsymbol{X}$ is given by the sum

$$X_d^+ = X_1 + \cdots + X_d.$$

Using a risk measure ρ, the aggregate random position X_d^+ is mapped into the real value $\rho(X_d^+)$, to be interpreted as the regulatory capital to be reserved in order to be able to safely hold $\boldsymbol{X}$. In this paper, we mainly deal with the case where ρ is a convex risk measure or the case where ρ is the Value-at-Risk (VaR). The evaluation of $\rho(X_d^+)$ is mainly a numerical issue once the joint distribution of $\boldsymbol{X}$ has been chosen or statistically evaluated. Estimating a multivariate distribution is a challenging task which is usually performed in two steps: first, d individual models F_j for the marginal loss exposures X_j are independently developed. Then, the marginal distributions are merged into a joint distribution using a dependence structure.

In fact, banks/insurance companies typically have better methods/more data for estimating a one-dimensional distribution for each risk type X_j than they have to estimate the overall dependence structure of $\boldsymbol{X}$. It is therefore reasonable to assume that the marginal distributions $F_1, \dots, F_d$ are known, while $F_{\boldsymbol{X}}$, the joint distribution of $\boldsymbol{X}$, varies in $\mathcal{F}_d(F_1, \dots, F_d)$, the so-called *Fréchet* class of all possible joint distributions having the fixed marginal models $F_1, \dots, F_d$. The choice of a single distribution in $\mathcal{F}_d(F_1, \dots, F_d)$ can lead to the miscalculation of the reserve $\rho(X_d^+)$. The implied model risk is referred to as *dependence uncertainty*.

A natural way to measure dependence uncertainty and, in more generality, model risk consists in finding the minimum and maximum possible values of the risk measure ρ evaluated over the class of candidate models; this is the approach taken in Cont (2006). In our framework, we define the smallest and biggest capitals to be held coherently with the given marginal distributions as

$$\underline{\rho}(X_d^+) = \inf\left\{\rho(X_d^+);\, F_{\boldsymbol{X}} \in \mathcal{F}_d(F_1, \dots, F_d)\right\}, \tag{1.1}$$

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and

$$\overline{\rho}(X_d^+) = \sup\left\{\rho(X_d^+); F_{\boldsymbol{X}} \in \mathcal{F}_d(F_1, \ldots, F_d)\right\}. \tag{1.2}$$

For any risk portfolio $(X_1, \ldots, X_d)$ having marginal distributions $F_1, \ldots, F_d$, it obviously holds that

$$\underline{\rho}(X_d^+) \leq \rho(X_d^+) \leq \overline{\rho}(X_d^+).$$

The difference $\overline{\rho}(X_d^+) - \underline{\rho}(X_d^+)$ is called *Dependence Uncertainty spread* (DU-spread) for ρ and is used to measure model uncertainty on the final capital reserve; see Embrechts et al. (forthcoming) for this terminology.

Computation of DU-spreads has been treated in the recent literature. The *analytical* computation of best- and worst-possible bounds on VaR can be performed only under some specific assumptions on the marginal distributions; see the survey paper Embrechts et al. (2014) for the state-of-the-art and a history of the problem. The analytical computation of worst-possible bounds on Expected Shortfall (ES) is in general straightforward, while for the best-case ES partial analytical results can be found in Wang and Wang (2011) and Bernard et al. (2014). For several classes of risk measures (including convex and distortion risk measures) Wang et al. (2014) provide a systematic way to compute the worst (and best) possible bounds across any homogeneous portfolio.

The *numerical* computation of DU-spreads of VaR and ES for arbitrary portfolios can be performed using the Rearrangement Algorithm described in Embrechts et al. (2013) (for the case of VaR) and in Puccetti (2013) (for ES) for dimensions d in the several hundreds or possibly thousands. Even if DU-spreads of VaR and ES are numerically available for practically any joint portfolio of risks, their relevance in actuarial practice has been recently questioned since they can be considerably large; see Aas and Puccetti (2014) for a real case study.

Therefore, in the recent literature many techniques to tighten DU-spreads were introduced. One possibility is to add extra (statistical) information on top of the knowledge of the marginal distributions. For instance, in Embrechts et al. (2013, Section 4) it is shown that having higher order (typically two-dimensional) marginals information on the joint portfolio leads to strongly improved bounds. The DU-spread of the VaR can be similarly reduced by specifying the copula on some subset of its domain (see Bernard et al., 2013a) or putting a variance constraint on the total position (see Bernard et al., 2013b).

In this paper we follow the idea to improve lower and upper risk bounds by introducing positive, respectively negative, dependence restrictions. Some results in this direction have been considered in the literature; see for instance Williamson and Downs (1990), Denuit et al. (1999), Embrechts et al. (2003), Rüschendorf (2005), Embrechts and Puccetti (2006) and Puccetti and Rüschendorf (2012). In particular, we show that positive dependence restrictions do not help to improve upper risk bounds essentially. They however allow to increase the lower risk bounds and therefore to reduce the model risk faced by an institution. Positive dependence information is introduced in Section 2 by the notions of orthant orders and weakly conditional increasing in sequence order. These orders are particularly capable to capture the concept of *stronger dependence* in the comparison of portfolios with fixed marginal distributions. In Section 3, we introduce a class of models which are bounded below in some stochastic ordering by a random vector whose marginals are decomposed in several independent subgroups with comonotonic dependence within the subgroups. This assumption allows to model a relevant variety of positive dependence restrictions.

We provide analytical upper and lower bounds on the VaR of the joint portfolio which are easily computable and are compared with the corresponding unconstrained bounds obtained without positive dependence assumptions. In Section 4, we deal with the case of law-invariant, convex risk measures, where we draw similar conclusions. While assuming a positive dependence structure typically improves the best-possible lower bound of a risk measure, it generally leaves unchanged the worst-possible upper risk bound. Finally, in Section 5 we discuss how *negative* dependence assumptions moderate also worst-case scenarios. We give a variety of applications of interest in quantitative risk management that can be easily adapted and closed formulas to be used in the risk management of real portfolios.

2. Dependence orders between risk vectors

In quantitative risk management, the components of a risk portfolio often have some positive dependence structure. A simple way to describe positive dependence is by using suitable stochastic orders between random vectors. In this section, we recall some natural positive dependence orders needed in the remaining part of the paper. For more details on these dependence notions we refer to Chapter 6 in Rüschendorf (2013). For a random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ in $\mathbb{R}^d$ we indicate with $F_{\boldsymbol{X}}$ its joint distribution function and with $\overline{F}_{\boldsymbol{X}}$ its survival function. Formally, for $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we denote

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = P(X_1 \leq x_1, \ldots, X_d \leq x_d),$$
$$\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) = P(X_1 > x_1, \ldots, X_d > x_d).$$

For two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathbb{R}^d$, we define

- the *upper orthant order* $\boldsymbol{Y} \leq_{\text{uo}} \boldsymbol{X}$, if $\overline{F}_{\boldsymbol{Y}}(\boldsymbol{x}) \leq \overline{F}_{\boldsymbol{X}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^d$;
- the *lower orthant order* $\boldsymbol{Y} \leq_{\text{lo}} \boldsymbol{X}$, if $F_{\boldsymbol{Y}}(\boldsymbol{x}) \leq F_{\boldsymbol{X}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^d$;
- the *concordance order* $\boldsymbol{Y} \leq_{\text{co}} \boldsymbol{X}$, if both $\boldsymbol{Y} \leq_{\text{uo}} \boldsymbol{X}$ and $\boldsymbol{Y} \leq_{\text{lo}} \boldsymbol{X}$ hold;
- the *weakly conditional increasing in sequence order* $\boldsymbol{Y} \leq_{\text{wcs}} \boldsymbol{X}$, if, for all $x \in \mathbb{R}$, all i with $1 \leq i \leq d-1$, and all component-wise increasing functions $f : \mathbb{R}^{d-i} \to \mathbb{R}$, we have

$$\begin{aligned}&\mathrm{Cov}(\mathbf{1}(Y_i > x), f(Y_{i+1}, \ldots, Y_d)) \\ &\quad \leq \mathrm{Cov}(\mathbf{1}(X_i > x), f(X_{i+1}, \ldots, X_d)).\end{aligned} \tag{2.1}$$

A random vector $\boldsymbol{Y}$ is smaller than $\boldsymbol{X}$ in the upper (lower) orthant order if the probabilities for upper (lower) orthants are ordered, i.e. the probability that all components jointly assume large (small) values is lower for $\boldsymbol{Y}$ than for $\boldsymbol{X}$. In the case $\boldsymbol{Y}$ has independent components with the same marginal distributions as $\boldsymbol{X}$ condition (2.1) is equivalent to the condition that

$$F_{(\boldsymbol{X}_{(i+1)}|X_i > x)} \geq_{st} F_{\boldsymbol{X}_{(i+1)}}, \quad \text{for all } i \leq d-1, x \in \mathbb{R}, \tag{2.2}$$

where $\boldsymbol{X}_{(i+1)} = (X_{i+1}, \ldots, X_d)$. The above condition means that the conditional distribution of $X_{(i+1)}$ given $X_i > x$ is stochastically larger than the distribution of $X_{(i+1)}$, for all real thresholds x. In this case condition (2.1) postulates that $\boldsymbol{X}$ is weakly associated in sequence, i.e WAS. This indicates some form of positive dependence of $\boldsymbol{X}$ and some consequences are described in Proposition 2.1 below. A slightly weaker dependence assumption than $\boldsymbol{Y} \leq_{\text{wcs}} \boldsymbol{X}$ is the supermodular order $\boldsymbol{Y} \leq_{\text{sm}} \boldsymbol{X}$ for random vectors having the same marginal distributions, $\boldsymbol{Y} \leq_{\text{wcs}} \boldsymbol{X}$ implies $\boldsymbol{Y} \leq_{\text{sm}} \boldsymbol{X}$; see Theorem 6.22 in Rüschendorf (2013). However, if compared to $\leq_{\text{sm}}$, the stochastic order $\leq_{\text{wcs}}$ is more easily interpretable and can be more easily checked in several functional models; see Rüschendorf (2004).

It is well known that in dimension $d = 2$ and assuming identical marginals for the two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ the four orders defined above are equivalent, i.e.

$$\boldsymbol{Y} \leq_{\text{uo}} \boldsymbol{X} \Leftrightarrow \boldsymbol{Y} \leq_{\text{lo}} \boldsymbol{X} \Leftrightarrow \boldsymbol{Y} \leq_{\text{co}} \boldsymbol{X} \Leftrightarrow \boldsymbol{Y} \leq_{\text{wcs}} \boldsymbol{X}.$$

The four orders are however different when $d \geq 3$, where we have that

$$\boldsymbol{Y} \leq_{\text{wcs}} \boldsymbol{X} \Rightarrow \boldsymbol{Y} \leq_{\text{co}} \boldsymbol{X} \Rightarrow \boldsymbol{Y} \leq_{\text{lo}} (\leq_{\text{uo}})\, \boldsymbol{X},$$

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