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Two parallel insurance lines with simultaneous arrivals and risks correlated with inter-arrival times

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ABSTRACT

We investigate an insurance risk model that consists of two reserves which receive income at fixed rates. Claims are being requested at random epochs from each reserve and the interclaim times are generally distributed. The two reserves are coupled in the sense that at a claim arrival epoch, claims are being requested from both reserves and the amounts requested are correlated. In addition, the claim amounts are correlated with the time elapsed since the previous claim arrival.

We focus on the probability that this bivariate reserve process survives indefinitely. The infinite-horizon survival problem is shown to be related to the problem of determining the equilibrium distribution of a random walk with vector-valued increments with 'reflecting' boundary. This reflected random walk is actually the waiting time process in a queueing system *dual* to the bivariate ruin process.

Under assumptions on the arrival process and the claim amounts, and using Wiener–Hopf factorization with one parameter, we explicitly determine the Laplace–Stieltjes transform of the survival function, c.q., the two-dimensional equilibrium waiting time distribution.

Finally, the bivariate transforms are evaluated for some examples, including for proportional reinsurance, and the bivariate ruin functions are numerically calculated using an efficient inversion scheme.

1. Introduction

We study a two-dimensional ruin problem for a bivariate risk reserve process in which claims are simultaneously requested from both reserves. The amounts of two simultaneously arriving claims may be correlated, and may also be correlated with the time elapsed since the previous claim arrival. Under assumptions on the arrival process and the claim sizes, we explicitly determine the Laplace–Stieltjes transform (LST) of the survival function.

Studies of *multidimensional* risk reserve processes are scarce in the insurance literature, although results about risk measures related to such models are highly relevant both from a theoretical and a practitioner's perspective. Multivariate ruin problems are relevant because they give insight into the behaviour of risk measures under various types of correlations between the insurance lines. One example is presented by multiple insurance lines within the same company which are interacting with each other as they evolve in time, via, say, coupled income rates. Another typical example is an umbrella type of insurance model, where a claim occurrence event generates multiple types of claims which may be correlated, and each type i claim is paid from its corresponding reserve $R^{(i)}$, such as car insurance together with health insurance or insurance against earthquakes. Yet another class of models is related to reinsurance problems, where a claim is shared between the insurer and one or more reinsurers.

In the existing risk and insurance literature, there are not many approaches towards analysing such complicated multidimensional models. A first attempt to assess multivariate risk measures can be found in the paper of Sundt (1999) about developing multivariate Panjer recursions which are then used to compute the distribution of the aggregate claim process, assuming simultaneous claim events and discrete claim sizes. Other approaches are deriving integro-differential equations for the various measures of risk and then iterating these equations to find numerical approximations (Chan et al., 2003; Gong et al., 2012), or computing bounds for the different types of ruin probabilities that can occur in a setting where more than one insurance line is considered (see Cai and Li, 2005 which considers multivariate phase-type claims). It is worth mentioning that very few papers (like Avram et al., 2008a; Badescu et al., 2011; Badila et al., 2014), analytically determine, e.g., the ruin probability for insurance models with more than one reserve.

In an attempt to solve the integro-differential equations that arise from such models, Chan et al. (2003) derive a so called 'boundary value problem' of a Riemann–Hilbert type for the bivariate

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Laplace transform of the joint survival function (see Badila et al., 2014 for details about such problems arising in the context of risk and queueing theory and the book (Cohen and Boxma, 1983) for an extended analysis of similar models in queueing). However, Chan et al. (2003) do not solve this functional equation for the Laplace transform. The law of the bivariate reserve process usually considered in the above mentioned works is that of a compound Poisson process with vector-valued jumps supported on the negative quadrant in $\mathbb{R}^2$, conditioned to start at some positive reserve, and linearly drifting along a direction vector that belongs to the positive quadrant. In Badila et al. (2014) a similar functional equation is taken as a departure point, and it is explained how one can find transforms of ruin related performance measures via solutions of the above mentioned boundary value problems. It is also shown that the boundary value problem has an *explicit* solution in terms of transforms, if the claim sizes are ordered. A special, important case is the setting of *proportional reinsurance*, which was studied in Avram et al. (2008a,b). There it is assumed that there is a single arrival process, and the claims are proportionally split among two reserves. In this case, the two-dimensional exit (ruin) problem becomes a *one*-dimensional first-passage problem above a piecewise linear barrier. Badescu et al. (2011) have extended this model by allowing a dedicated arrival stream of claims into only one of the insurance lines. They show that the transform of the time to ruin of at least one of the reserve processes can be derived using similar ideas as in Avram et al. (2008a).

The approach we take in this paper generalizes ideas in Badila et al. (2013, 2014), and will allow us to extend those two studies. In Section 3 we derive a similar functional equation as in Badila et al. (2014) for the survival function related to a 2-dimensional reserve process, but unlike Badila et al. (2014) we do not assume that the claim intervals are exponentially distributed. Furthermore we assume the claim size vector to be correlated with the time elapsed since the previous arrival. Such a correlation is quite natural; e.g., a claim event that generates very large claims could be subjected to additional administrative/regulatory delays. The type of correlation between the inter-arrival time and the vector of claim sizes is an extension to two dimensions of the dependence structure studied in Badila et al. (2013) for a generalized Sparre-Andersen model. It involves making a rationality assumption regarding the trivariate LST of inter-arrival time and claim size vector (Assumption 2.1), which extends the case where the vector with the aforementioned components has a multivariate phase type distribution (MPH). In addition, we also make the assumption that the claim sizes are a.s. ordered (Assumption 2.2). Under these assumptions, we obtain our main result: An explicit expression for the (LST of the) two-dimensional survival function, for a large class of vectors of interclaim times and claim amounts of both reserves.

The paper is organized in the following way. In Section 2 we describe the model and present the main assumptions we will be working with. Section 3 is dedicated to some general theory for random walks in the plane; we show a useful relation between the two-dimensional risk reserve process and a version of the random walk which has the boundary of the nonnegative quadrant in $\mathbb{R}^2$ as an impenetrable barrier. This relation also makes it clear that determining the survival function is equivalent with determining the two-dimensional waiting time distribution in a dual two-queue two-server queueing model with simultaneous arrivals of customers at both queues. With the help of the random walk/queueing process we derive, in Section 4, a stochastic recursion for the LST of the finite horizon survival function. In Section 5 we resolve the stationary version of this stochastic recursion, Formula (5.1). The key tool used is a one-parameter Wiener–Hopf factorization of the bivariate kernel appearing in Eq. (5.1). More precisely, the Wiener–Hopf factors will depend on one parameter, which is the first argument of that bivariate kernel; see Proposition 2 in Section 5. The main result, Theorem 2, gives the LST of the survival function, or equivalently the stationary distribution of the waiting time/reflected random walk inside the positive quadrant (see also Remark 3).

In Section 6 we explain how to calculate the transform obtained in Theorem 2 for some examples, and we numerically calculate the ruin functions/waiting time distributions using an efficient inversion algorithm of den Iseger (2006). Finally we also point out that the numerical results suggest that the ruin functions appear to be stochastically ordered for various types of correlations between inter-arrival times and claim sizes, a positive correlation leading to smaller ruin probabilities.

2. Model description

Let us begin with the general assumptions on the two reserves. The reserves start with non-negative initial capital $(u^{(1)}, u^{(2)})$; as long as there are no arrivals, the reserves increase linearly with positive rates $(c^{(1)}, c^{(2)})$. At the nth claim arrival epoch, claim sizes $B_n^{(1)}$ and $B_n^{(2)}$ are respectively requested from each reserve. The time between the $(n-1)$th and nth claim arrival is denoted by A_n. The sequence $\{A_n, B_n^{(1)}, B_n^{(2)}\}_{n\geq 1}$, is assumed to be an i.i.d. sequence, but within a triple, $(A_n, B_n^{(1)}, B_n^{(2)})$ are allowed to be correlated. We will use $A, B^{(1)}, B^{(2)}$ respectively for the generic inter-arrival time and claim sizes. In the above-described very general set-up, the following assumption will allow us to explicitly determine the ruin/survival probabilities by using Wiener–Hopf factorization:

Assumption 2.1 (*On the Joint Transform of* $\mathbf{A}, \mathbf{B}^{(1)}, \mathbf{B}^{(2)}$)**.** The triple transform

$$H(q_0, q_1, q_2) := \mathbb{E}e^{-q_0 A - q_1 B^{(1)} - q_2 B^{(2)}} \tag{2.1}$$

is a rational function in q_i, $i = 0, 1, 2$, i.e., it has representation $\frac{N(q_0,q_1,q_2)}{D(q_0,q_1,q_2)}$ such that $N(q_0, q_1, q_2)$ and $D(q_0, q_1, q_2)$ are polynomials in the variables q_i, $i = 0, 1, 2$.

$N(q_0, q_1, q_2)$ and $D(q_0, q_1, q_2)$ must satisfy some conditions, because their ratio is a transform, such as,

$$\lim_{|q_0|\to\infty,\ \mathcal{R}e\, q_0>0} H(q_0, q_1, q_2) = \mathbb{E}[e^{-q_1 B^{(1)} - q_2 B^{(2)}} 1_{\{A=0\}}],$$

with $1_{\{E\}}$ the indicator function of event E. We can assume without loss of generality that $A > 0$ a.s. Because of the above limit, this means the degree of N as a polynomial in q_0, $N_{q_1,q_2}(q_0)$, is strictly less than the degree of D as a polynomial in q_0: $D_{q_1,q_2}(q_0)$, for all q_1 and q_2.

Remark 1. The class of rational multivariate Laplace–Stieltjes transforms contains the class of LSTs of multivariate Phase-Type distributions (MPH); see Bladt and Nielsen (2010), where the rational transform class is called multivariate matrix-exponential (MME). All of the well-known classes of multivariate Phase-Type distributions are MME. Actually, all the examples we will present are MPH distributions with a specific correlation structure which are a special case of Kulkarni's MPH* class (Kulkarni, 1989). There is no point in restricting ourselves to any of these subclasses. We will fully exploit the algebraic representation of rational functions in order to derive explicitly the transforms of the two-dimensional survival/ruin functions.

The reserve process $\bar{R}_t = (\bar{R}_t^{(1)}, \bar{R}_t^{(2)})$ evolves as

$$\bar{R}_t = \bar{u} + ct - \sum_{i=1}^{n(t)} B_i, \quad \text{where } \bar{u} := (\bar{u}^{(1)}, \bar{u}^{(2)}),\ c := (c^{(1)}, c^{(2)}),$$

$$B_i := (B_i^{(1)}, B_i^{(2)}),$$