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On optimal reinsurance policy with distortion risk measures and premiums

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ABSTRACT

In this paper, we consider the problem of optimal reinsurance design, when the risk is measured by a distortion risk measure and the premium is given by a distortion risk premium. First, we show how the optimal reinsurance design for the ceding company, the reinsurance company and the social planner can be formulated in the same way. Second, by introducing the "marginal indemnification functions", we characterize the optimal reinsurance contracts. We show that, for an optimal policy, the associated marginal indemnification function only takes the values zero and one. We will see how the roles of the market preferences and premiums and that of the total risk are separated.

1. Introduction

The problem of optimal reinsurance design lies at the heart of reinsurance studies. A reinsurance policy is a contract, according to which part of the risk of an insurance company (the ceding company) is transferred to another insurance company (the reinsurance company), in exchange for receiving a premium. Different reinsurance contracts have been introduced in the reinsurance market among which the quota-share, stop-loss, stop-loss after quota-share and quota-share after stop-loss have received more attention due to their appealing optimality properties. Borch (1960) (and also Arrow, 1963) showed that, subject to a budget constraint, the stop-loss policy is an optimal reinsurance contract for the ceding company when the risk is measured by variance (or by a utility function). Recent extensions of the same problem have been studied in Kaluszka (2001), Young (1999) and Kaluszka and Okolewski (2008).

The problem of optimal reinsurance design has been studied by using risk measures and risk premiums, due to their development and application in finance and insurance. For instance, in a framework where the ceding company's risk is measured either by Value at Risk (VaR) or Conditional Tail Expectation (CTE),[1] with the Expected Value Premium Principle as the risk premium, Cai and Tan (2007) found the optimal retention levels. Later, in the same framework, Cai et al. (2008) showed that the stop-loss and the quota-share are the most optimal reinsurance contracts. In Bernard and Tian (2009) also, the authors have considered optimal risk management strategies of an insurance company subject to regulatory constraints when the risk is measured by VaR and CVaR. In recent years, researchers have tried to extend the optimal reinsurance design problem to larger families of risk measures and risk premiums. For instance, Cheung (2010) and Chi and Tan (2013) have extended the problem by using a family of general risk premiums; in these two papers the risk of the ceding company is measured either by VaR or CTE. On the other hand, Cheung et al. (2014) have extended the problem by using general law-invariant convex risk measures, whereas the risk premium is considered to be the Expected Value Premium Principle. In the existing literature, either only the family of risk measures or only the family of risk premiums is extended, while in many applications it is desirable to extend both at the same time.

The present paper, considers a framework which extends, at the same time, the set of risk measures and the risk premiums to the family of distortion risk measures and premiums. First, we show that in this framework the ceding, the reinsurance and the social planner problems can be formulated in the same way. Second, we characterize the optimal solutions by introducing the notion of marginal indemnification function. A marginal indemnification function is the marginal rate of changes in the value of a reinsurance contract. We show that any optimal solution to the reinsurance problem has a marginal indemnification function which only takes the values zero and one. Remarkably, we can separate the roles of the market preferences and premiums and that of the total risk are separated. Finally, we have to point out that, by using a very simple fact that any Lipschitz continuous function has a derivative that is bounded by its Lipschitz constant, we introduce a useful technique in this paper that can generalize many already existing results to the distortion risk measures and risk premiums.

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[1] It can be shown that for continuous distributions, CTE is equal to the Conditional Value at Risk (CVaR), which will be introduced later in this paper.

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It is worth mentioning that the families of distortion risk measures and risk premiums contain very important particular cases; for instance, the family of the co-monotone sub-additive law invariant coherent risk measures (Kusuoka, 2001), the family of generalized spectral risk measures (Cont et al., 2010), the generalized distortion measures of risk (Wang, 1995), Wang's risk premiums (Wang, 1995; Wang et al., 1997), Expected Value Premium Principle and many others.

The rest of the paper is organized as follows: Section 2 introduces the mathematical notions and notations that we use in this paper. In Section 3 the general set-up of the ceding, the reinsurance and the social planner problems will be presented. In Section 4 the results on characterizing the optimal reinsurance contracts will be presented. In Section 5 we provide some corollaries and examples.

2. Preliminaries and notations

Let $(\Omega, P, \mathcal{F})$ be a probability space, where Ω is the "states of the nature", P is the physical probability measure and $\mathcal{F}$ is the σ-field of measurable subsets of Ω. The set of all random variables on Ω is denoted by L^0. In this paper, we consider only two periods of time, 0 and T, where 0 represents the beginning of the year, when a contract is written, and T represents the end of the year, when liabilities are settled. Every random variable represents losses at time T. For any $X \in L^0$, the cumulative distribution function associated with X is denoted by F_X.

2.1. Distortion risk measures

Let $\Pi : [0, 1] \to [0, 1]$ be a non-decreasing function such that $\Pi(0) = 1 - \Pi(1) = 0$. Let us introduce the set $\mathcal{D}_\Pi$ as follows:

$$\mathcal{D}_\Pi = \left\{ X \in L^0 \mid \int_0^1 \mathrm{VaR}_t(X) d\Pi(t) \in \mathbb{R} \right\}, \tag{1}$$

where the integral above is the Lebesgue integral with respect to the measure induced by Π on $[0, 1]$ and

$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} | P(X > x) \le 1 - \alpha\}, \quad \alpha \in [0, 1].$

Let $\mathcal{X} \subseteq \mathcal{D}_\Pi$ be a set of loss random variables. A distortion risk measure ϱ_Π (or simply ϱ) is a mapping from $\mathcal{X}$ to $\mathbb{R}$ defined as

$$\varrho_\Pi(X) = \int_0^1 \mathrm{VaR}_t(X) d\Pi(t). \tag{2}$$

By introducing $g(x) := 1 - \Pi(1-x)$, one can see that the distortion form (2) can be represented in the form of a Choquet integral

$$\varrho(X) = \int_{-\infty}^0 (g(S_X(t)) - 1)\, dt + \int_0^\infty g(S_X(t)) dt, \tag{3}$$

where $S_X = 1 - F_X$ is the survival function associated with X. In the literature, g is known as the distortion function. The idea of the definition of a distortion risk measure goes back to the axiomatic definition of risk premiums for insurance policies Wang et al. (1997).

Note that the definition of $\mathcal{D}_\Pi$ helps us to better deal with optimization problem related to distortion risk measures. For instance, it is clear that if $X \in \mathcal{D}_\Pi$ and $X \ge 0$, then $\left\{Y \in L^0 | 0 \le Y \le X\right\} \subseteq \mathcal{D}_\Pi$. In the following, we will see that if X denotes the random values of the aggregate claims for an insurance company, then all the optimization problems that we will deal with are in the set $\left\{Y \in L^0 | 0 \le Y \le X\right\}$.

A popular example of a distortion risk measure is Value at Risk, introduced earlier, where $\Pi(t) = 1_{[\alpha,1]}(t)$. A Conditional Value at Risk (CVaR) is a distortion risk measure whose distortion function is given by $\Pi(t) = \frac{t-\alpha}{1-\alpha} 1_{[\alpha,1]}(t)$, and can be represented as

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_t(X) dt. \tag{4}$$

The family of spectral risk measures which was introduced first in Acerbi (2002), has the same representation as (2), where Π is also convex. One can readily see that ϱ_Π is positive homogeneity, translation invariant, monotone, law invariance and comonotonic additive. It can be shown that all law-invariant co-monotone additive coherent risk measures can be represented as (2); see Kusuoka (2001). A risk measure in the form (2) is important from different perspectives. First of all it makes a link between the risk measures theory and the behavioral finance as the form (2) is a particular form of Choquet utility. Second, (2) contains a family of risk measures which are statistically robust. In Cont et al. (2010) it is shown that a risk measure $\varrho(x) = \int_0^1 \mathrm{VaR}_t(x) d\Pi(t)$ is robust if and only if the support of $\varphi = \frac{d\Pi(t)}{dt}$ (the derivative is in general a distribution and not a function) is away from zero and one. For example Value at Risk is a risk measure with this property.

For more reading on distortion risk measures one can see Sereda et al. (2010), Wu and Zhou (2006), Balbás et al. (2009) and Wang et al. (1997).

2.2. Distortion risk premiums

A risk premium in general is introduced as a continuous mapping on a subset of L^0 of loss random variables, which maps any loss variable to a number representing its premium. A general definition for the risk premium in the literature is proposed by Wang et al. (1997) in an axiomatic manner. Wang et al. (1997) characterize the family of cash invariant, positive homogeneous, co-monotone additive risk premiums which satisfy the following continuity property

$$\lim_{d\to\infty} \pi(X \wedge d) = \pi(X) \quad \text{and} \quad \lim_{d\to 0} \pi((X-d)_+) = \pi(X),$$

as

$$\pi(X) = \int_0^\infty g(S_X(t)) dt, \tag{5}$$

where $g : [0, 1] \to [0, 1]$ is a non-decreasing function such that $g(0) = 0$ and $g(1) = 1$. When the function g is convex the premium is called Wang's Premium Principle.

By similar change of variable for risk measures (i.e., $\Pi(x) = 1 - g(1-x)$), the following equality holds for a premium represented in (5)

$$\pi(X) = \int_0^1 \mathrm{VaR}_t(X) d\Pi(t). \tag{6}$$

Definition 1. Let $\Pi : [0, 1] \to [0, 1]$ be a non-decreasing function such that $\Pi(0) = \Pi(1) - 1 = 0$. The distortion premium π_Π is introduced on a set $\mathcal{X} \subseteq \mathcal{D}_\Pi$ of loss random variables as

$$\pi_\Pi(X) = \int_0^1 \mathrm{VaR}_t(X) d\Pi(t). \tag{7}$$

A popular example of a distortion risk premium is Wang's premium[2] introduced by the following distortion function (reminding that $\Pi(x) = 1 - g(1-x)$) known as Wang's transformation

$$g_\beta(x) = \Phi(\Phi^{-1}(x) + \beta), \tag{8}$$

where $\beta \in \mathbb{R}$ is a real number and Φ is the CDF of the normal distribution with the mean equal to zero and the standard deviation equal to one.

3. Problem set-up

In this section, we set up the optimal reinsurance design problem for the ceding company, the reinsurance company and the so-

[2] This premium was first introduced by Wang, however in general if Π is convex we also call a distortion premium a Wang premium.