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The time of deducting fees for variable annuities under the state-dependent fee structure

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ABSTRACT

We investigate the total time of deducting fees for variable annuities with state-dependent fee. This fee charging method is studied recently by Bernard et al. (2014) and Delong (2014) in which the fees deducted from the policyholder's account depend on the account value. However, both of them have not considered the problem of analyzing probabilistic properties of the total time of deducting fees. We approximate the maturity of a general variable annuity contract by combinations of exponential distributions which are (weakly) dense in the space that is composed of all probability distributions on the positive axis. Working under general jump diffusion process, we derive analytic formulas for the expectation of the time of deducting fees as well as its Laplace transform.

1. Introduction

Variable Annuities (VAs) are generally issued with minimum guarantee on the death or maturity benefits and keep in high demand due to the embedded guarantees. Nowadays, a variety of guarantees are provided, such as Guaranteed Minimum Death Benefits (GMDBs), Guaranteed Minimum Maturity Benefits (GMMBs), Guaranteed Minimum Accumulation Benefits (GMABs) and so on, see Bauer et al. (2008). In the academic circle, VAs have received substantial attention, see among others, Lee (2003), Gerber and Shiu (2003), Ko et al. (2010), Bacinello et al. (2011) and Gerber et al. (2012, 2013).

In general, insurers charge expenses for the provision of the guaranteed benefits from the policyholder's account by a fixed rate, and most actuarial literatures assume that the fee rate is fixed as well, see e.g. Hyndman and Wenger (2014). However, this fixed fee structure has several disadvantages which have been noted by Bernard et al. (2014) and Delong (2014). For example, as the guaranteed benefits embedded in variable annuities are similar to put options, if the account value is high, the guarantees are (deep) out-of-the-money. In this case, a higher deducted fee will yield incentives for policyholders to lapse the policy, see Bauer et al. (2008).

Recently, Bernard et al. (2014) put forward a dynamic fee structure. In their paper, the fees are deducted at a fixed rate only if the account value is lower than a pre-specified level. They have derived formulas for calculating the fee rates under the Black–Scholes model. Delong (2014) then extended their model to an incomplete financial market which is made up of two risk assets that are modeled by a two-dimensional Lévy process, and he considered a general state-dependent fee which is determined by a function of the account value. In his paper, the fee for the guaranteed benefit can be computed by solving an equation, and a strategy for hedging the guarantee is also characterized.

Under a state-dependent fee structure, sometimes insurers do not have any fees income, e.g., when the account value is higher than a pre-specified level in Bernard et al. (2014). Therefore, insurers are interested in the problem that how long they can collect fees or alternatively, how long they cannot. In nature, insurers are interested in the probabilistic properties of the total time of collecting fee, for example its expectation. Besides, this total time will determine insurers' income directly and have a significant impact on the account value indirectly. The above problem has not been considered in Bernard et al. (2014) and Delong (2014), and we are the first to investigate it to our knowledge.

In this paper, we assume that the fees are deducted as in Bernard et al. (2014), which is one of the fee charging structures investigated in Delong (2014). Under the simple Black–Scholes model as well as the complex hyper-exponential jump diffusion process model, we obtain analytic formulas for the Laplace transform of the total time of deducting fees and its expectation as well.

The rest of this paper is organized as follows. Our model is introduced in Section 2. In Section 3, under the hyper-exponential jump diffusion process, we derive our main results. In Section 4, three special cases of the general model are discussed in detail

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where the formulas become simpler. Finally, some conclusions are given in Section 5.

2. The model

Suppose $X = (X_t)_{t\geq 0}$ is a jump diffusion process, i.e.,

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{k=1}^{N_t} Z_k, \tag{2.1}$$

where μ and X_0 are constants; $\{W_t;\ t \geq 0\}$ is a standard Brownian motion with $W_0 = 0$, and $\sigma > 0$ is the volatility of the diffusion; $\{Z_k;\ k = 1, 2, \ldots\}$ are independent and identically distributed random variables supported on $\mathbb{R}/\{0\}$, and the common probability density function (pdf) of $\{Z_k;\ k = 1, 2, \ldots\}$ is denoted by $f_Z(z)$; $\{N_t;\ t \geq 0\}$ is a Poisson process with rate λ; moreover, $\{W_t\}$, $\{N_t\}$ and $\{Z_k;\ k = 1, 2, \ldots\}$ are independent. The law of X starting from x is denoted by $\mathbb{P}_x$ and $\mathbb{E}_x$ represents the corresponding expectation, and for the sake of brevity, we write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

We define the process $S = (S_t)_{t\geq 0}$

$$S_t = S_0 e^{X_t - X_0}, \tag{2.2}$$

which can be explained as the value of one unity of the reference fund or stock underlying the variable annuity contract, and it is reasonable to assume that $\mathbb{E}[X_1] > 0$. According to (2.6) in Delong (2014), if we let F_t denote the policyholder's account value at time t, then its dynamics are represented by the following stochastic differential equation (SDE)

$$dF_t = F_{t-}\frac{dS_t}{S_{t-}} - g(F_{t-})dt, \quad t > 0, \tag{2.3}$$

and F_0 is the initial premium invested by the insured, and g is a function which represents a state-dependent fee charged by the insurer.

In Bernard et al. (2014), they assumed that X is a Brownian motion and g is given by

$$g(x) = \alpha x \mathbf{1}_{\{x<B\}}, \tag{2.4}$$

where α and B denote the deduction fee rate and a pre-specified level, respectively; and $\mathbf{1}_A$ is the indicator function of a set A. Function (2.4) means the expense is charged only when the account value is less than the pre-specified level B. In this paper, we consider the case that g satisfies Eq. (2.4) and the investigation of general g is left for future research. For the advantages of this fee deducting method, see Bernard et al. (2014) and Delong (2014).

Let U_t satisfy the following SDE

$$dU_t = dX_t - \alpha \mathbf{1}_{\{U_t<b\}}dt, \quad t > 0, \quad \text{and} \quad U_0 = X_0, \tag{2.5}$$

where $b = \ln\left(\frac{B}{F_0}\right)$. Then, when $U_0 = 0$, we obtain the following equation of F_t and U_t by Itô's formula.

$$F_t = F_0 e^{U_t}, \quad t \geq 0. \tag{2.6}$$

Because the parameter $\sigma > 0$ and the jump part of X is a compound Poisson process, we have the following lemma of the existence of a unique strong solution of (2.5). For the proof of it, one can see Remarks 2 and 3 in Kyprianou and Loeffen (2010) and the proof of Lemma 2.1 in Delong (2014) for reference.

Lemma 2.1. *Eq. (2.5) exists a unique strong solution. Moreover the solution U is a strong Markov process.*

Consider a customer age z purchasing a VA contract. If the guarantee embedded is GMMBs, then the maturity of the contract is a given date T. For the guarantee of GMDBs, the maturity is given by $\min\{T_z, T\}$ for endowment policy or T_z for whole life (see Gerber et al., 2012 for example), where T_z is the random variable representing the time of death of the insured and independent of the account value process F_t. In this paper, we ignore other reasons of terminating the contract, e.g., the customer lapses the policy.

We first consider the case that the maturity is T_z. Under the fee structure (2.3) and (2.4), the insurer is interested in the expectation of the total time of charging fees:

$$\mathbb{E}\left[\int_0^{T_z} \mathbf{1}_{\{U_t<b\}}dt\right] = \int_0^{\infty} \mathbb{E}\left[\int_0^t \mathbf{1}_{\{U_s<b\}}ds\right] f_{T_z}(t)dt, \tag{2.7}$$

where $f_{T_z}(t)$ denotes the probability density function of T_z.

Remark 2.1. To evaluate (2.7), the value of α in (2.5) is required to be specified first. In principle, the value of α is determined by risk neutral pricing rule for some martingale measure. Here, we are more interested in obtaining the probability characteristic of time of deducting fee. Therefore, in this paper, we assume that the parameter α is exogenous and focus on the calculation of (2.7) with given α.

From Ko and Ng (2007) or Dufresne (2007), we obtain the following result: in the space of all probability distribution on the positive axis, the subset of linear combinations of exponential distributions is (weakly) dense. Therefore, we can approximate $f_{T_z}(t)$ by $\sum_i c_i f_{T_i}(t)$, where $f_{T_i}(t)$ is the density function for some exponential distributions. Hence, the problem of calculating (2.7) reduces to compute the following expectation:

$$\mathbb{E}\left[\int_0^{\tau} \mathbf{1}_{\{U_t<b\}}dt\right], \tag{2.8}$$

where τ is an exponential random variable which is independent of the process U.

For a contract with maturity date T, we first note that (2.8) is related to the Laplace transform of $\mathbb{E}\left[\int_0^T \mathbf{1}_{\{U_s<b\}}ds\right]$. Therefore, if the contract expires at a fixed time T, the total time of deducting fees can also be obtained from (2.8) through taking inverse Laplace transform for instance. In addition, if the maturity is $\min\{T_z, T\}$, then we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\min\{T_z,T\}} \mathbf{1}_{\{U_s<b\}}ds\right] \\
&= \mathbb{E}\left[\int_0^{T_z} \mathbf{1}_{\{U_s<b\}}ds \mathbf{1}_{\{T_z<T\}}\right] + \mathbb{E}\left[\int_0^{T} \mathbf{1}_{\{U_s<b\}}ds \mathbf{1}_{\{T_z\geq T\}}\right] \\
&= \mathbb{P}(T_z < T)\int_0^{+\infty} \mathbb{E}\left[\int_0^t \mathbf{1}_{\{U_s<b\}}ds\right] g_{T_z|T_z<T}(t)dt \\
&\quad + \mathbb{P}(T_z \geq T)\mathbb{E}\left[\int_0^T \mathbf{1}_{\{U_s<b\}}ds\right],
\end{aligned} \tag{2.9}$$

where $g_{T_z|T_z<T}(t)$ is the conditional density function of T_z on the event of $\{T_z < T\}$. Similar to the above, we can use linear combinations of exponential distributions to approximate the density function $g_{T_z|T_z<T}(t)$. Hence, the computation of the first term on the right-hand side of (2.9) also reduces to calculate (2.8).

In short, to calculate the total time of deducting fees for a general VA contract, the most important quantity is (2.8). In this paper, we assume that the density function of τ in (2.8) is given by

$$\nu e^{-\nu t}, \quad t > 0, \tag{2.10}$$

where $\nu > 0$ is a constant. Of course, it is very difficult (or even impossible) to calculate (2.8) for arbitrary pdf $f_Z(z)$. In this paper, we assume that $f_Z(z)$ is given by

$$f_Z(z) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i z}\mathbf{1}_{\{z>0\}} + \sum_{j=1}^{n} q_j \vartheta_j e^{\vartheta_j z}\mathbf{1}_{\{z<0\}}, \tag{2.11}$$

where $p_i > 0$, $\eta_i > 0$ for all $i = 1, \ldots, m$; $q_j > 0$, $\vartheta_j > 0$ for all $j = 1, \ldots, n$; $\sum_{i=1}^m p_i + \sum_{j=1}^n q_j = 1$, $\eta_1 < \eta_2 < \cdots < \eta_m$ and $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_n$.

Remark 2.2. The process X defined by (2.1) and (2.11) is called hyper-exponential jump diffusion process (HEP). The HEP has sev-