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Robust investment–reinsurance optimization with multiscale stochastic volatility☆

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ABSTRACT

This paper investigates the investment and reinsurance problem in the presence of stochastic volatility for an ambiguity-averse insurer (AAI) with a general concave utility function. The AAI concerns about model uncertainty and seeks for an optimal robust decision. We consider a Brownian motion with drift for the surplus of the AAI who invests in a risky asset following a multiscale stochastic volatility (SV) model. We formulate the robust optimal investment and reinsurance problem for a general class of utility functions under a general SV model. Applying perturbation techniques to the Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation associated with our problem, we derive an investment–reinsurance strategy that well approximates the optimal strategy of the robust optimization problem under a multiscale SV model. We also provide a practical strategy that requires no tracking of volatility factors. Numerical study is conducted to demonstrate the practical use of theoretical results and to draw economic interpretations from the robust decision rules.

1. Introduction

Many risk and insurance problems can be formulated as a stochastic control problem. For instance, the allocation of an insurer's reserves can be viewed as a stochastic control problem of maximizing an objective function which specifies the balance between profit and risk, see Yong and Zhou (1999). Two major financial activities for insurers to achieve this goal are: entering a reinsurance contract to transfer its risks to other firms, and investing in the risk-free and risky financial assets. Such problems were studied under different objectives such as minimizing the ruin probability (Promislow and Young, 2005); optimization with a VaR constraint (Chen et al., 2010); maximizing the utility with no-shorting constraint (Bai and Guo, 2008); investigating the Hamilton–Jacobi–Bellman (HJB) equation of related problems (Cao and Wan, 2009); incorporating general insurance (Liu and Ma, 2009); and considering the mean–variance objectives in Zeng and Li (2011) and Chen and Yam (2013).

Apart from a certain objective function and a known stochastic model, recent advances take the ambiguity aversion, uncertainty associated with the model and the risk aversion, into account as suggested by the Ellsberg paradox. Knight (1921) points out that ambiguity is distinct from the familiar notion of risk but subtle. Ambiguity and risk aversion are distinct factors in explaining the insurer's behaviors. We are interested in maximizing the anticipated utility of the terminal wealth of the insurer with the concerns of ambiguity and risk aversion, by adopting the notion of robust portfolio optimization. In fact, the robust control theory has been applied to the investment–reinsurance (IR) problems. For example, Zhang and Siu (2009) study the robust IR problem via a max–min approach. Robust asset allocation problem via a penalty max–min approach is studied by Maenhout (2004) in accordance with the robust decision rules in Anderson et al. (1999). Yi et al. (2013) investigate the IR problem with model uncertainty under the Heston stochastic volatility (SV) model. However, these analysis only focus on a specific family of utility functions: power utility and exponential utility. We aim to extend existing results to a general class of concave utility functions under SV models, and offer implementable solution to the case of multi-scale SV model using asymptotic theory.

In this paper, we consider the surplus process of the insurer following a Brownian motion with drift by adopting the framework of Promislow and Young (2005). Investment can be made between a risky asset and a risk-free money market account, where the risky asset can be interpreted as the market index. This consideration facilitates the implementation because we use option data to infer the volatility surface to enhance investment decision. Usually,

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index option data are large enough for model calibration purpose. Although our analysis applies to different stochastic models for the risky asset price, we concentrate on a fixed surplus process for the insurer. Therefore, we formulate the robust IR problem for a general utility, analogously to Maenhout (2004), under a general SV model for the risky asset.

We derive an asymptotic solution to the robust IR problem under the multiscale SV model, which contains a fast time scale factor and a slow time scale factor. These models and the related perturbation techniques are described in Fouque et al. (2011). The effect of multiscale SV on dynamic fund protection is studied by Wong and Chan (2007). One advantage of using the multiscale SV model is that it allows us to infer the parameters by calibrating to market information. Moreover, the implementation is simple and accurate. In portfolio optimization, Fouque et al. (2015) obtain the asymptotic solution for the nonlinear Merton problem for the general utility. This motivates us to extend their results to the robust IR problem with general utility functions. In addition, the market information can be effectively incorporated into the optimal strategy under this framework. To make the presentation comprehensible, we illustrate the formulation and the derivation with the fast mean-reverting SV (FMRSV) model (i.e. only the fast time scale factor is considered) in the main context. The corresponding results to multiscale (or multifactor) SV model are collected in the Appendix because the additional effort for the derivation is minimal.

A key advantage of our approach is its application to general utility functions. To show this advantage, we use the mixture of power utilities as an example as this utility function produces a nonlinear risk-tolerances and a non-constant relative risk aversion. The empirical studies in Brunnermeier and Nagel (2008) document the relevancy of the time-varying relative risk aversion in practical decision making process.

The remainder of this paper is organized as follows. Section 2 presents the formulation of the robust IR problem with a general utility function under general SV models. We then asymptotically solve the robust IR problem under the FMRSV model in Section 3. A practical portfolio strategy is proposed in Section 4 so that no tracking of the instantaneous volatility is required. Section 5 uses numerical studies to examine the impact of SV factor in the robust IR problem and the performance of the strategy under the mixture of power utilities. We also address the implication of the robustness in our formulation. Section 6 concludes.

2. Problem formulation

2.1. The reference model

Consider the continuous-time surplus process with reinsurance and investment opportunities. The reference model is defined over the physical measure $\mathbb{P}$. Following Promislow and Young (2005), the claim process C of the insurer assumed as

$$dC(t) = adt - bdW_t^G,$$

where $a, b > 0$ are rate of the claim and the volatility of the claim process, respectively, and W_t^G is the standard $\mathbb{P}$-Brownian motion. To make the model more appealing, we further assume that the ratio a/b is large enough ($a/b > 3$) such that the probability of realizing a negative claim is small in any period of time. When the reinsurance strategy is absent in the analysis, the insurance premium rate is $\varsigma_0 = (1+\tau)a$ with the safety loading $\tau > 0$ implies the surplus process G_0 as

$$dG_0(t) = \varsigma_0 dt - dC(t) = \tau adt + bdW_t^G.$$

When reinsurance is allowed, the insurer can divert a proportion of all premiums to another insurer (reinsurer) to manage the insurance risk. Let $1-q(t)$ be the reinsurance fraction at time t where $q(t) \in [0, +\infty)$. The process $\{q(t)\}_{t\in[0,T]}$ is called a reinsurance strategy. When $q(t) > 1$, the underlying insurer itself offers reinsurance service to other insurers. When $q(t) \in [0, 1]$, the insurer makes proportional reinsurance. In this case, the fraction $1-q(t)$ of each claim is paid by the counterparty reinsurer. When the reinsurance premium rate $\varsigma_1 = (1+\eta)(1-q(t))a$ with safety loading $\eta \geq \tau > 0$ is charged as the expense for reducing the potential risk, the surplus process G with the reinsurance strategy $q(t)$ becomes,

$$\begin{aligned} dG(t) &= \varsigma_0 dt - q(t)dC(t) - \varsigma_1 dt \\ &= [\lambda + \eta q(t)]adt + bq(t)dW_t^G, \end{aligned} \tag{1}$$

where $\lambda = \tau - \eta \leq 0$.

In addition, the underlying insurer can invest in a risky asset and a risk-free asset. We postulate the price of the risky asset S to follow an Itô process with SV driven by Y:

$$\begin{cases} dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t^S, \\ dY_t = m(Y_t)dt + \alpha(Y_t)[\rho dW_t^S + \bar{\rho} dW_t^Y], \end{cases} \tag{2}$$

where $\bar{\rho} = \sqrt{1-\rho^2}$, and W_t^S and W_t^Y are independent standard $\mathbb{P}$-Brownian motions, while W_t^G is independent of W_t^S and W_t^Y.

The insurer determines her wealth X allocation between the risky and risk-free assets and the reinsurance strategy. We use $l(t)$ to denote the amount of wealth in the risky asset at time t and the remaining amount in risk-free asset at rate r. The reinsurance strategy $q(t)$ and investment strategy $l(t)$ then form the IR strategy pair $\pi = (q, l)'$. The principle of continuous-time self-financing trading yields the following dynamics for the wealth process X:

$$\begin{aligned} dX_t &= dG_t + \frac{l}{S_t}dS_t + r(X_t - l)dt \\ &= [a\lambda + a\eta q + (\mu(Y_t) - r)l + rX_t]dt \\ &\quad + bqdW_t^G + \sigma(Y_t)ldW_t^S. \end{aligned} \tag{3}$$

2.2. Robust stochastic control problem

Classical approaches aim at maximizing the anticipated utility of the terminal wealth with the fixed investment horizon $T < \infty$:

$$\sup_{\pi\in\Pi} \mathbb{E}^{\mathbb{P}}[U(X_T)], \tag{4}$$

where Π is the set of admissible strategies π. But we are interested in incorporating the ambiguity aversion into the problem for an ambiguity-averse insurer.

Our approach stems on the belief that the insurer has some confidence in the reference measure $\mathbb{P}$ and is willing to consider a class of possible measures $\mathcal{Q}$, which are "similar" to $\mathbb{P}$. To clarify the meaning of "similar" there, we employ the concept of equivalent measures, analogous to Anderson et al. (1999). Specifically, alternative measures are induced by a class of probability measures equivalent to $\mathbb{P}$: $\mathcal{Q} := \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P}\}$. By the Girsanov theorem, for each $\mathbb{Q} \in \mathcal{Q}$, there is a stochastic process $\varphi^{\mathbb{Q}}(t) = (\varphi_G^{\mathbb{Q}}(t), \varphi_S^{\mathbb{Q}}(t), \varphi_Y^{\mathbb{Q}}(t))'$, which can be regarded as the model misspecification factors, such that

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \nu(t) = \exp\left(\int_0^t \varphi^{\mathbb{Q}}(s)'dW_s - \frac{1}{2}\int_0^t \varphi^{\mathbb{Q}}(s)'\varphi^{\mathbb{Q}}(s)ds\right),$$

where $\mathcal{F}_t = \sigma(\{W_s\}_{0\leq s\leq t})$ and $W_t = (W_t^G, W_t^S, W_t^Y)'$. Moreover, if $\varphi^{\mathbb{Q}}(t)$ satisfies the Novikov condition,

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \varphi^{\mathbb{Q}}(s)'\varphi^{\mathbb{Q}}(s)ds\right)\right] < \infty,$$

then the process $\nu(t)$ is a positive $\mathbb{P}$-martingale and $\widetilde{W}_t := (\widetilde{W}_t^G, \widetilde{W}_t^S, \widetilde{W}_t^Y)$ becomes a $\mathbb{Q}$-Brownian motion in $\mathbb{R}^3$, where $d\widetilde{W}_t =$