Journal of Economic Dynamics & Control 37 (2013) 231–250

Contents lists available at SciVerse ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

An inverse finite element method for pricing American options

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A R T I C L E I N F O

Article history:
Received 9 December 2010
Received in revised form
24 January 2012
Accepted 27 July 2012
Available online 8 August 2012

JEL classification:
G130

Keywords:
Inverse finite elements
Convergence analysis
American options
Black–Scholes model

A B S T R A C T

The pricing of American options has been widely acknowledged as "a much more intriguing" problem in financial engineering. In this paper, a "convergency-proved" IFE (inverse finite element) approach is introduced to the field of financial engineering to price American options for the first time. Without involving any linearization process at all, the current approach deals with the nonlinearity of the pricing problem through an "inverse" approach. Numerical results show that the IFE approach is quite accurate and efficient, and can be easily extended to multi-asset or stochastic volatility pricing problems. The key contribution of this paper to the literature is that we have managed to provide a comprehensive convergence analysis for the IFE approach, including not only an error estimate of the adopted discrete scheme but also the convergence of the adopted iterative scheme, which ensures that our numerical solution does indeed converge to the exact one of the original nonlinear system.

1. Introduction

As is well known, one of the major topics in today's quantitative finance research is the valuation of financial derivatives, such as options. For quite a long time, it has been widely acknowledged that pricing American options is a "much more intriguing" problem (see Huang et al., 1996; Ju, 1998; Longstaff and Schwartz, 2001), whose challenge mainly stems from the nonlinearity originated from the inherent characteristic that an American option can be exercised at any time during its lifespan. This additional right of being able to exercise the option early, in comparison with a European option, casts the American option pricing problem into a free boundary problem, which is highly nonlinear and far more difficult to deal with. Since most traded stock and commodity options in today's financial markets are of American style, it is important to ensure that American-style securities can be priced accurately as well as efficiently.

Recently, there is a breakthrough in the pricing of American options as an analytical closed-form pricing formula was successfully derived by Zhu (2006a). Although this formula has a great significance on the theoretical side of option pricing, it is not computationally appealing, as the formula involves two infinite sums of infinite double integrals, which take a formidable amount of time to evaluate. Till now, approximation methods are still popular among those market practitioners as they are usually faster with acceptable accuracy.

In the literature, of all the approximation methods, there are predominately two types, analytical approximations and numerical methods for the valuation of an American option contract. Typical methods in the first category include the compound-option approximation method (Geske and Johnson, 1984), the quadratic approximation method (Barone-Adesi and Whaley, 1987; MacMillan, 1986), the randomization approach (Carr, 1997), the integral-equation method (Carr et al., 1992;

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http://dx.doi.org/10.1016/j.jedc.2012.08.002

Chiarella et al., 1999; Jacka, 1991; Kim, 1990), and the Laplace transform method (Zhu, 2006b). However, as just has been pointed out by Wu and Kwok (1997), the generalization of these quasi-analytical approaches to some exotic options may not be easy. Moreover, it seems quite difficult to extend all these methods to high-dimensional problems. On the other hand, the numerical methods for the valuation of American options typically include the FD (finite difference) method (Brennan and Schwartz, 1977; Schwartz, 1965; Wu and Kwok, 1997; Zhu and Zhang, 2011), the FE (finite element) method (Allegretto et al., 2002), the radius basis function method (Hon and Mao, 1997), the binomial tree method (Cox et al., 1979), the moving boundary approach (Muthuraman, 2008), the Monte Carlo simulation technique (García, 2002; Grant et al., 1996), and the least square approach (Longstaff and Schwartz, 2001).

The numerical methods proposed in the literature can be further categorized into two types, according to the different ways of dealing with the nonlinearity associated with American options. The methods in the first category usually adopt some sort of iterative methods to solve the discretized nonlinear system. For example, Elliott and Ockendon (1982) discretized a general class of free boundary problems by the FE method, and then solved the resulting system of nonlinear algebraic equations with an iterative approach. Fang and Oosterlee (2008) solved the pricing problem of European options by replacing the transitional probability density function by its Fourier-cosine series expansion. Later on, they extended this scheme to the pricing of Bermudan options, by adding a main loop based on the Newton iterations to determine the early exercise points (Fang and Oosterlee, 2009). Of course, one may view that a scheme designed for pricing Bermudan options can be adopted to price an American option, at least from numerical point of view. But, a fundamental difference is that the former requires the eligible exercise dates are a subset of grids, whereas there is no such requirement for the latter, which enables the selection of grids in a way to maximize numerical efficiency for a given numerical accuracy. Furthermore, a pity of this elegant approach presented by Fang and Oosteree is that they only concentrated on analyzing the convergence of the discrete solution of the system of nonlinear algebraic equations to the exact one of the original nonlinear system, not mentioning the convergence of the adopted iterative methods at all, which should also form an essential component of the complete proof of the convergence of a numerical scheme, because without the guarantee of the convergence of the adopted iterative scheme, the numerical value may still fail to converge to the exact solution of the original problem.

On the other hand, the methods in the second category usually begin with linearization process to deal with the nonlinear nature of the American option pricing problems. A typical example in this category is the multi-level FD approach proposed by Wu and Kwok (1997). In their approach, a linearization process is firstly invoked, and the information obtained will be used to start the second-phase computation for the option prices. Whilst appealing, there was no theoretical proof for the convergency of this elegant approach, and moreover, even if the convergency could be established theoretically, it would only guarantee that the numerical results converge to the linearized system, rather than the original nonlinear one. Although most authors demonstrated the convergency through their numerical examples, lack of convergency proof requires cautions when these approaches are adopted on a case-by-case basis. This has motivated us to explore a "convergency-proved" approach for the valuation of American options.

In this paper, the IFE (inverse finite element) approach (see Alexandrou, 1989), which was originated in solving nonlinear problems associated with phase change, is introduced to the field of option pricing for the first time. As its name suggests, this method is based on an inversely defined problem, and is implemented by using a combination of both the FE method and the Newton iteration scheme. Essentially, the IFE approach solves the location (the nodes of the finite elements), at which the variable has a specific value. The advantage of the IFE approach mainly comes from two aspects. Computationally, this method is quite efficient since it allows the use of full Newton iteration method with its inherent quadratic convergence. Furthermore, the IFE method does not require any complicated co-ordinate transformations, in which the location of the free boundary is somehow fixed in the new co-ordinate system, and thus has a great potential of being extended to pricing American options based on multi-factor models (e.g., using stochastic volatility models) or even pricing American options written on multi-assets.

Comparing with some of the widely used numerical methods for solving American option prices (Wu and Kwok, 1997; Zhu and Zhang, 2011), the IFE approach involves no linearization process at all, while most of the others require some sort of linearization of the PDE (partial differential equation) system. It is this rather unique feature of the IFE approach that has made it worthwhile to pursue a theoretical convergence analysis, since such an analysis can ensure that the numerical solution does indeed converge exactly to the original nonlinear PDE system, contrary to those involving linearization process, with which the analysis, if achieved, can only guarantee that the numerical results will converge to the linearized system rather than the original nonlinear one. On the other hand, it is usually easy to design a numerical scheme to solve a PDE system, but much harder to prove the convergency of the scheme. It is probably even more difficult to provide a theoretical proof for the convergency of the IFE approach since this method involves both the FE formulation and the Newton iteration as part of the solution procedure. In other words, as far as the completeness of the convergence proof of the IFE method is concerned, it should consist of not only the convergence of the approximate solution to one of the original nonlinear system, but also that of the adopted Newton iterative scheme. It is for this reason that the issue of the convergency of the IFE approach was not previously addressed at all. One of the important contributions of this paper is that we have managed to provide not only an error estimate of the FE discretization, but also the convergence of the adopted iteration scheme, which forms an indispensable ingredient of the complete convergence analysis of the entire IFE scheme adopted to price American options.

The paper is organized as follows. In Section 2, we introduce the PDE system that the price of an American put option must satisfy under the BS (Black–Scholes) model. In Section 3, we present the IFE approach in detail. In Section 4, we provide a