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A fast approach to analysis and optimization of viscoelastic beams

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ABSTRACT

A new truly-mixed finite element for the analysis of viscoelastic beams is presented that is based on the additive decomposition of the bending moment in a viscoelastic and a purely elastic contribution. Bending moments are the primary variables that belong to $H^2(0,\ell)$ whereas the kinematic variables (that are the velocities and not the displacements as usual) are globally discontinuous and elementwise linear. As for the peculiarities of the proposed finite element, results from relaxation and creep numerical tests are presented in much detail and a quadratic convergence assessed for all the variables involved. In the second part of the paper, a fast approach to structural (sizing) optimization, set as a topology optimization problem, of such viscoelastic beams is presented in the presence of time-dependent objective functions. Within a gradient-based minimization scheme that is solved via the method of moving asymptotes (Svanberg, 1987), a dual sensitivity analysis approach is derived and representative numerical results presented and discussed in much detail.

1. Introduction

Viscoelasticity is a constitutive feature of materials that finds applications in several areas such as structural engineering (concrete modeling), biomedics (soft tissues) and also synthetic materials and polymers fabrication [2].

From a modeling viewpoint, the most classic approach to viscoelasticity relies on the hereditary integral formulation that is based on the interpretation of viscoelastic materials as materials with memory: the current stress may be computed as a time integral of the strain history [3]. Most times such viscoelastic behavior is introduced by using a complex definition of the Young modulus where the imaginary part is responsible for the viscous part of the response, see [4,5] among others. A different approach has been used in [6] that models the response of 2D viscoelastic media using a mixed finite element framework that neither adopts the hereditary formulation nor the complex stiffness approach. It is conversely based on the additive decomposition of the total stress into a purely elastic and a viscoelastic contribution and, as to kinematics, on the adoption of the velocity as opposed to the displacement that is by far the most classical choice. The main advantage of such an approach is that the resulting governing equations neither exhibit complex coefficients nor call for the computation of integrals of hereditary nature, say of convolution type, that may lead to cumbersome algorithms for the solution. The first objective of the present paper is then to propose a fast approach that following the streamline proposed in [6] allows the computation of the response of beams made of viscoelastic materials. For simplicity sake the proposed approach is concerned with the thin (Kirchhoff) kinematic model.

As to optimal design of viscoleastic materials and structures, a by no means exhaustive list of appealing contributions is reported next. Paper [7] is concerned with determining composite materials layouts capable of exhibiting high damping in a viscoelastic regime, [5,8] deal with material optimization at the microstructural level so as to obtain materials and structures of prescribed (and usually extreme) homogeneized properties, [4] applies a topology optimization strategy to the design of periodic composites capable of maximum attenuation of propagating waves, [9] deals with the optimization of non-classically damped linear dynamic structures in the presence of harmonic external loads, which should be considered a peculiarity of the paper as opposed to most of the existing literature on the subject that deals with eigenvalue optimization wherein loads are simply neglected. From the computational standpoint of optimization methods for viscoelastic (and more generally time-dependent systems), a crucial step is represented by the semi-analytic computation of the gradient of the optimization function as proposed in [10,11]. A similar approach is proposed in the present paper even though the evolution of the viscoelastic system is herein governed by a first-order differential–algebraic equation as opposed to [10,11] that consider second-order differential equations and this remarkable difference causes our approach for computing sensitivities to be different

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though inspired by the same philosophy. Such semi-analytic approach for the computation of the gradient of the objective function represents one of the key ingredients for the structural (sizing) optimization method of viscoelastic beams that is presented in the second part of the paper.

The paper is organized as follows. Section 2 presents the continuous variational formulation of the viscoelastic beam under investigation whose mixed finite-element space-discretization scheme is outlined in Section 3. Cubic shape functions are adopted for the element-wise approximation of the bending moment that globally belongs to $H^2(0,\ell)$ whereas velocities are element-wise linear and globally discontinuous. The time-discretization of the resulting differential–algebraic equation is tackled in Section 4 where the adoption of the two-step algorithm proposed in [12] and applied in [6] is suggested. The truly-mixed finite-element approach proposed herein shows some peculiar aspects as to the imposition of boundary conditions at extreme and/or intermediate sections of the beam that are elucidated in Section 5. The optimization problem is presented in Section 6 along with the algorithm for the semi-analytic computation of the gradient of the objective function whereas numerical studies concerning the assessment of the convergence order of the proposed finite-element approximation scheme, applications to creep and relaxation tests, and representative results concerning the optimization of viscoelastic beams are left to Section 7.

2. Truly-mixed variational formulation

Truly-mixed variational formulations are based on mixed stress-displacement approximations of Hellinger–Reissner type wherein stresses are the regular primary variables whereas displacements play the role of globally discontinuous Lagrange multipliers [13]. One of the peculiarities of the proposed formulation is that the kinematic variables are not displacements and curvatures but their time derivatives.

The phenomenological model of Fig. 1 is used to express the adopted viscoelastic constitutive law in terms of bending moments M and dual curvatures χ that is based on the additive decomposition of the total bending moment M as

$$M = M^0 + M^1, \tag{1}$$

where M^0 and M^1 are the bending moments that may be attributed to the viscoelastic component labeled as "0" and the purely elastic one labeled as "1" in Fig. 1, respectively. The constitutive law of the viscoelastic phenomenological model therefore reads

$$\begin{cases} \frac{1}{EJ_E^0}\dot{M}_0 + \frac{1}{EJ_V^0}M_0 = \dot{\chi}_V^0 + \dot{\chi}_E^0 = \dot{\chi} \\ \frac{1}{EJ_E^1}\dot{M}_1 = \dot{\chi}_E^1 = \dot{\chi} \end{cases}. \tag{2}$$

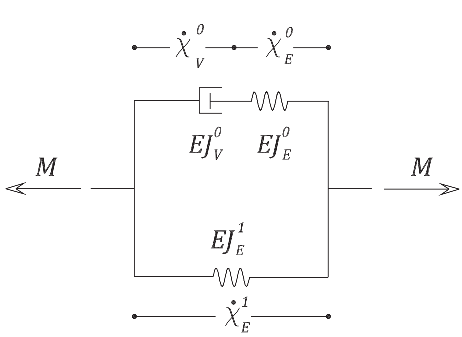

Fig. 1. Standard solid phenomenological model.

Upon denoting with q the load per unit length, thanks to Eq. (1), the thin beam equilibrium equation reads

$$M'' = M^{0\,\prime\prime} + M^{1\,\prime\prime} = -q. \tag{3}$$

Finally, if v denotes the time derivative of the transverse displacement, i.e. the velocity field, the compatibility equation reads

$$-v'' = \dot{\chi}, \tag{4}$$

where, as usual, superposed dots and primes indicate time and space differentiation, respectively.

By equating the right hand sides of Eqs. (2) and (4), the time-derivative of the curvature $\dot{\chi}$ may be eliminated and, after some algebra that includes two integration by parts, one arrives at the following Hellinger–Reissner truly-mixed formulation:

Find $(M_0, M_1, v) \in (H_0^2 \times H_0^2 \times L^2)$ such that:

$$\begin{cases} \int_0^\ell \frac{1}{EJ_E^0}\dot{M}_0 M_0^* + \int_0^\ell \frac{1}{EJ_V^0} M_0 M_0^* + \int_0^\ell v M_0^{*\,\prime\prime} = 0 \\ \int_0^\ell \frac{1}{EJ_E^1}\dot{M}_1 M_1^* + \int_0^\ell v M_1^{*\,\prime\prime} = 0 \\ \int_0^\ell M_0'' w + \int_0^\ell M_1'' w = -\int_0^\ell q w \end{cases} \tag{5}$$
$\forall M_0^* \in H_0^2, \forall M_1^* \in H_0^2, \forall w \in L^2.$

For future use, with a slight abuse of notations, Eq. (5) is rearranged as follows. Let

$$\boldsymbol{Y} = \begin{pmatrix} M_0 \\ M_1 \\ v \end{pmatrix} \tag{6}$$

be the vector grouping the unknown fields to be computed,

$$\boldsymbol{Q} = \begin{pmatrix} 0 \\ 0 \\ -\int_0^\ell q w \end{pmatrix} \tag{7}$$

the right-hand side accounting for the external action,

$$\boldsymbol{D}_1 = \begin{pmatrix} \int_0^\ell \frac{1}{EJ_E^0} M_0 M_0^* & 0 & 0 \\ 0 & \int_0^\ell \frac{1}{EJ_E^1} M_1 M_1^* & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

and

$$\boldsymbol{D}_0 = \begin{pmatrix} \int_0^\ell \frac{1}{EJ_V^0} M_0 M_0^* & 0 & \int_0^\ell v M_0^{*\,\prime\prime} \\ 0 & 0 & \int_0^\ell v M_1^{*\,\prime\prime} \\ \int_0^\ell M_0'' w & \int_0^\ell M_1'' w & 0 \end{pmatrix}, \tag{9}$$

or, in short,

$$\boldsymbol{D}_1 = \begin{pmatrix} A_E^0 & 0 & 0 \\ 0 & A_E^1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{D}_0 = \begin{pmatrix} A_V^0 & 0 & B \\ 0 & 0 & B \\ B^T & B^T & 0 \end{pmatrix}, \tag{10}$$

the structural matrices. Eq. (5) may then be formally rewritten as

$$\boldsymbol{D}_1 \dot{\boldsymbol{Y}}(t) + \boldsymbol{D}_0 \boldsymbol{Y}(t) = \boldsymbol{Q}(t). \tag{11}$$

that turns out to be a Differential–Algebraic Equation (DAE) that calls for suitable integration algorithms as detailed in a forthcoming section.

3. Space finite-element discretization

As to the discrete variational formulation, cubic Hermite polynomials $\phi_i(x), i = 1, \ldots, 4$ are used for the bending moments M and M^* so as to get global C^1 continuity, whereas the velocities v