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Coincidence of the Shapley value with other solutions satisfying covariance

Koji Yokote [a,*], Yukihiko Funaki [a], Yoshio Kamijo [b]

[a] School of Political Science and Economics, Waseda University, 1-6-1, Nishi-Waseda, Shinjuku-ku, Tokyo 169-8050, Japan
[b] Kochi University of Technology, School of Economics and Management, 2-22 Eikokuji, Kochi City, Kochi 780-8515, Japan

HIGHLIGHTS

- We analyze the coincidence between the Shapley value and other solutions in TU game.
- We analyze the coincidence with the prenucleolus, the CIS and ENSC values.
- We identify the necessary and sufficient condition for the coincidence.
- We identify the coincidence condition in three well-known allocation problems.

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ABSTRACT

We identify the necessary and sufficient condition under which the Shapley value coincides with the prenucleolus for general TU games. For 0-normalized 3-person games, the coincidence holds if and only if the game is symmetric or satisfies the PS property (Kar et al., 2009). We also identify the necessary and sufficient coincidence condition in the following allocation problems: the airport games (Littlechild and Owen, 1973), the bidder collusion games (Graham et al., 1990) and the polluted river games (Ni and Wang, 2007). The coincidence between the Shapley value and the CIS and ENSC values is discussed as well.

1. Introduction

One of the main goals in the cooperative game theory is to understand the relationship among various solutions. In previous studies, axiomatic characterization of solutions has played a central role in achieving this goal. Another major approach is to provide conditions under which different solutions coincide, i.e., assign the same payoff vector. This approach has two advantages. First, coincidence conditions help us understand the similarity among solutions from a geometric point of view. Second, coincidence conditions inform us that a payoff vector is supported by two different solutions, thereby strengthening the payoff vector as a desirable outcome.

In previous studies, the coincidence between the Shapley value (Shapley, 1953) and the prenucleolus (Schmeidler, 1969) has been intensively discussed. Kar et al. (2009) introduced a new sufficient condition for the coincidence, called the PS property. This property states that "the sum of a player's marginal contribution to any coalition S and its complement coalition $N \setminus (S \cup \{i\})$ is a player specific constant" (Kar et al., 2009). Chun and Hokari (2007) proved that the Shapley value coincides with the prenucleolus in the class of 2-games, which is a subclass of the games satisfying the PS property. Chang and Tseng (2011) extended the PS property to a more general class of games. Chun et al. (2016) proved that the Shapley value coincides with the prenucleolus in the appointment problems by using the PS property.

The purpose of this paper is to identify the necessary and sufficient condition under which the Shapley value coincides with other solutions satisfying covariance. We first identify the condition under which the Shapley value coincides with the prenucleolus for general TU games by using the basis introduced by Yokote et al. (2016). Then, we apply the condition to specific classes of games and obtain clearer results. We prove that, for 0-normalized 3-person games, the Shapley value coincides with the prenucleolus if and only if the game is symmetric or satisfies the PS property. We also identify the necessary and sufficient coincidence condition in the following allocation problems: the airport games (Littlechild and Owen, 1973), the bidder collusion games (Graham et al., 1990) and the polluted river games (Ni and Wang, 2007).

* Corresponding author.
E-mail addresses: sidehand@toki.waseda.jp (K. Yokote), funaki@waseda.jp (Y. Funaki).

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To compare coincidence conditions between different solutions, we also identify the necessary and sufficient condition under which the Shapley value coincides with the CIS and ENSC values (Driessen and Funaki, 1991).

The reminder of this paper is organized as follows. Section 2 presents notations and definitions. In Section 3, we revisit the commander games by Yokote et al. (2016) and identify the necessary and sufficient coincidence condition for general TU games. In Section 4, we identify the necessary and sufficient condition in the three allocation problems. Section 5 provides concluding remarks. All proofs are provided in Section 6.

2. Notations and definitions

2.1. TU games

Let $N = \{1, \ldots, n\}$, $n \geq 2$, be a finite set of players. We call $S \subseteq N$ a coalition of N. A characteristic function $v : 2^N \to \mathbb{R}$ assigns a real number to each coalition of N; we assume $v(\emptyset) = 0$. We call $v(S)$ the worth of coalition S. A pair (N, v) is called a (TU)-game. The set of all games with player set N is denoted as Γ^N. For $(N, v) \in \Gamma^N$, we abuse notation and simply write v.[1]

For any $v, w \in \Gamma^N$ and $\alpha \in \mathbb{R}$, we define addition and scalar multiplication as follows: $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$, $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$. Then, we can identify Γ^N as a linear space $\mathbb{R}^{2^n-1}$.

A game $v \in \Gamma^N$ is called simple if $v(S) = 0$ or 1 for all $S \subseteq N$. A game $v \in \Gamma^N$ is called convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. A game is called symmetric if the worth of coalitions depends only on the number of players. For any game $v \in \Gamma^N$ and $\beta \in \mathbb{R}^n$, we define $v + \beta \in \Gamma^N$ by $(v + \beta)(S) = v(S) + \sum_{i \in S} \beta_i$ for all $S \subseteq N, S \neq \emptyset$. For any $v \in \Gamma^N$, we define the 0-normalized game v^0 by $v^0(S) = v(S) - \sum_{i \in S} v(\{i\})$ for all $S \subseteq N, S \neq \emptyset$.

Let $v \in \Gamma^N$, $S \subseteq N$, $S \neq \emptyset$, and $x \in \mathbb{R}^n$. We define the excess of coalition S with respect to x in the game v by $e(S, x, v) = v(S) - \sum_{i \in S} x_i$.

2.2. Solutions and axioms

For any $v \in \Gamma^N$, we define the preimputation set $PI(v)$ as follows:

$$PI(v) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \right\}.$$

A solution assigns an element of $PI(v)$ to each game $v \in \Gamma^N$. We define five solutions. The Shapley value (Shapley, 1953) is defined as follows: for any $v \in \Gamma^N$,

$$\phi_i(v) = \sum_{S \subseteq N : i \in S} \frac{(n - |S|)!(|S| - 1)!}{n!} \big(v(S) - v(S \setminus \{i\}) \big) \text{ for all } i \in N.$$

For any $x, y \in \mathbb{R}^n$, $y \geq_{lex} x$ means that y is greater than x in the lexicographic ordering of $\mathbb{R}^n$. Let $\theta(x) = (\theta_1(x), \theta_2(x), \ldots, \theta_{2^n-1}(x)) \in \mathbb{R}^{2^n-1}$ denote the sequence of excess of $S \subseteq N, S \neq \emptyset$, with respect to x, where $\theta_t(x) \geq \theta_{t+1}(x)$ for all t, $1 \leq t \leq 2^n - 2$. The nucleolus, introduced by Schmeidler (1969), is defined as follows: for any $v \in \Gamma^N$,

$$Nu(v) = \{x \in I(v) : \theta(y) \geq_{lex} \theta(x) \text{ for all } y \in I(v)\},$$

where

$$I(v) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N \right\}.$$

[1] In Section 6.3, we do not follow this rule and explicitly refer to the player set by writing (N, v).

The prenucleolus is defined as follows:

$$\eta(v) = \{x \in PI(v) : \theta(y) \geq_{lex} \theta(x) \text{ for all } y \in PI(v)\}.$$

It is known that the nucleolus coincides with the prenucleolus in any convex game. We define the CIS and ENSC values introduced by Driessen and Funaki (1991): for any $v \in \Gamma^N$,

$$CIS_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n} \text{ for all } i \in N,$$

$$ENSC_i(v) = v(N) - v(N \setminus \{i\}) + \frac{v(N) - \sum_{j \in N} \big(v(N) - v(N \setminus \{j\}) \big)}{n} \text{ for all } i \in N.$$

We list up axioms satisfied by a solution ψ.

Covariance (COV) For any $v \in \Gamma^N$ and $\beta \in \mathbb{R}^n$, $\psi(v + \beta) = \psi(v) + \beta$.

Efficiency (EFF) For any $v \in \Gamma^N$, $\sum_{i \in N} \psi_i(v) = v(N)$.

Equal Treatment Property (ETP) Let $v \in \Gamma^N$ and $i, j \in N$. If $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\}$, then $\psi_i(v) = \psi_j(v)$.

If a solution ψ satisfies the above three axioms, then for any 2-person game $v \in \Gamma^N$, ψ coincides with the standard solution defined as follows[2] :

$$\psi_i(v) = v(i) + \frac{v(N) - v(i) - v(j)}{2} \text{ for all } i, j \in N, i \neq j.$$

Since all the solutions defined in this section satisfy the three axioms, they coincide in the class of 2-person games. This coincidence, however, does not hold in general. In what follows, we focus on games with more than 2 players and identify the condition under which different solutions coincide.

3. Commander games and coincidence conditions

Throughout this section, we fix a player set N. We refer to the commander games introduced by Yokote et al. (2016). Then, we apply the games to the analysis of the coincidence of solutions.

For each $T \subseteq N$, $T \neq \emptyset$, we define the T-commander game $\bar{u}_T$ by

$$\bar{u}_T(S) = \begin{cases} 1 \text{ if } |S \cap T| = 1, \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 1 of Yokote et al. (2016). *The set $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$ is a basis of Γ^N.*

Thus, any game $v \in \Gamma^N$ can be expressed by a linear combination of $\{\bar{u}_T\}_{\emptyset \neq T \subseteq N}$. Let $d(T, v)$ denote the coefficient of $\bar{u}_T$ in the linear combination, i.e.,

$$v = \sum_{T \subseteq N : T \neq \emptyset} d(T, v) \bar{u}_T. \quad (1)$$

Proposition 1 of Yokote et al. (2016). *For any $v \in \Gamma^N$,*

$$d(\{i\}, v) = \phi_i(v). \quad (2)$$

Namely, the coefficients related to singletons coincide with the Shapley value.

For any $v \in \Gamma^N$, let v^{Sh} denote the game $v - \phi(v)$. Then, v^{Sh} satisfies

$$v^{Sh}(S) = v(S) - \sum_{i \in S} \phi_i(v) = e(S, \phi(v), v) \text{ for all } S \subseteq N, S \neq \emptyset. \quad (3)$$

[2] See Lemma 5.4.3 of Peleg and Sudhölter (2007).