Mathematical Social Sciences 89 (2017) 10–19

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/mss

The Owen and Shapley spatial power indices: A comparison and a generalization

Mathieu Martin [a], Zephirin Nganmeni [a], Bertrand Tchantcho [b,*]

[a] Department of Economics, THEMA, Cergy Pontoise, France
[b] Department of Mathematics, ENS Yaoundé, Cameroon and THEMA, Cergy Pontoise, France

HIGHLIGHTS

- We propose a generalization of Owen index.
- We provide a formal proof that the Shapley power index extends the Owen power index.
- We generalize the distance approach of Owen's for spatial games.
- We study the link between our generalization and the Shapley power index.

ARTICLE INFO

Article history:
Received 6 November 2014
Received in revised form 22 February 2017
Accepted 7 May 2017
Available online 31 May 2017

ABSTRACT

Spatial games take into account the position of any voter in the space. In this class of games, two main indices of political power were defined. The first by Owen (1971) and the second, by Shapley (1977), later on extended in a two-dimensional space by Owen and Shapley (1989). We propose a generalization of Owen index. We show that the method proposed by this later in which players ordering is based on the distance between bliss and political issues points, yields the Shapley index if issues can be any point in the space.

1. Introduction

In 1971, Owen proposed a modification of the Shapley–Shubik power index by taking into account the fact that due to personal affinities or ideological differences among the players, certain coalitions are more easily formed than the others. This means that unlike Shapley–Shubik power index case, all the orderings of players do not have the same probability to occur. In order to be able to talk about affinities, similarities among players as well as differences, they are represented in the finite-dimensional Euclidean space $\mathbb{R}^m$. Each individual (or player) is identified by a position in the space, called his *ideal or bliss* point. Owen assumes that the issues and the players bliss points belong to a hypersphere and the order in which the players favor each issue is given by the Euclidean distance between the issue and the bliss point. The power of a player is the proportion of the hypersphere such that he is pivotal. In a different approach, Shapley assumes that issues are political directions, bliss points are any point in the space and the order of the players at each issue is determined by the orthogonal projections of their bliss point on the issue. The power of a player is the proportion of directions for which this player is pivotal.

Owen (1971) and Shapley (1977) are the two seminal papers that generalize the classical Shapley and Shubik (1954) index in a spatial environment.[1] The first application of these two indices to the distribution of power in a real political institution can be found in Frank and Shapley (1981). They use the voting records of the nine-members voting body of the United States Supreme Court justice to estimate their ideological positions, and they apply these attitude-dependent indices to estimate the distribution of power in the Supreme Court. Many other theoretical developments and applications can be found in the literature: see Shenoy (1982), Rapoport and Golan (1985), Rabinowitz and McDonald (1986), Grofman et al. (1987), Shapley and Owen (1989), Feld and Grofman (1990), Ono (1996), Aleskerov (2008), Godfrey et al. (2011), Alonso-Meijide et al. (2011), Barr and Passarelli (2009) or Benati and Vittucci Marzetti (2013) among others.

In this paper we focus on Owen (1971) and Shapley (1977) and give a formal proof that the Shapley power index extends the Owen power index. In Shapley's approach, political issues are directions

* Corresponding author.; fax: +33 1 34 25 62 33
E-mail address: btchantcho@yahoo.fr (B. Tchantcho).

[1] For a clear and simple presentation of the Shapley–Shubik power index and its spatial approach, see, e.g., Straffin (1994).

http://dx.doi.org/10.1016/j.mathsocsci.2017.05.003

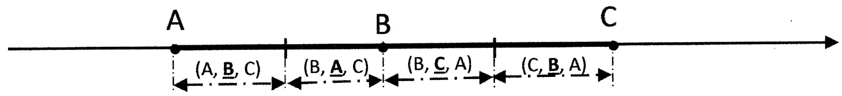

Fig. 1. Distance approach on a segment.

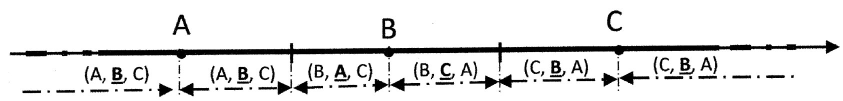

Fig. 2. Distance approach on a line.

of the space. It is well known that there is a perfect isomorphism between directions and points of an m-sphere. The set of feasible bliss points is the whole space. The restriction of Shapley power index on the hypersphere clearly leads to the Owen power index. Our main contribution is the generalization of the distance approach of Owen's original index by relaxing the assumption that the issues and the ideal points belong to a hypersphere, and we study the link between our generalization and the Shapley power index. Indeed, the restriction of Owen on feasible issues and bliss points means that, for instance in a two-dimensional context, they are located on a circle which is a considerable constraint. In many cases, the set of political issues can take any form. For example, for public good location problems, the domain is usually countable or finite, eventually with several constraints (urban planning constraints, environmental constraints, accessibility constraints, etc.) it can also be finite. In our model, there is no restriction on the set of feasible issues. The order of players is still determined by the distance between the bliss point and the political issue. The power of a player is the proportion of points for which he is pivotal. We show that, when there is no restriction on the domain, (that is, all issues over all the considered ideological dimensions have the same probability to occur), our generalization coincides with the Shapley power index. It turns out that even though in the definition of Shapley's power index orderings of players are generated by orthogonal projection to the issues, it can also be seen as distance-based orderings. Our findings prove that the calculation method based on distances thus originates a sufficiently general index, whose interpretation is rather intuitive. Our methodology comes directly from Owen (1971). In order to understand clearly our approach, consider the simple example with three individuals A, B and C on a classical left–right axis where simple majority is used. There are two issues in considering Shapley's approach: from left to right and from right to left. In the two cases, B is pivotal and then, the distribution of power is 0 for A and C and 1 for B. Consider now a distance approach (B is the midpoint of [AC]) and the Fig. 1.

The first segment with the sequence $\left(A, \underline{\mathbf{B}}, C\right)$ above means that for any point of this segment B is pivotal (B is second in terms of distance). The distribution of power is then 1/4 for A and C and 1/2 for B. The result is quite different from the one obtained with Shapley. Consider now the same graph except that we replace the segment by a line and the following graph (see Fig. 2).

Clearly, the new distribution of power is 0 for A and C and 1 for B, as with Shapley. Our purpose is to generalize this example for any number of dimensions.

Two main drawbacks of the famous Shapley technique were raised by Passarelli and Barr (2007): (1) the excessive concentration of power measures and (2) the too high sensitivity to players' location in the ideological space. They argued that the probability of coalition formation depends on (i) how close players are in terms of their attitudes, and (ii) how likely different issues are to come up for a vote. In Owen (1971) and Shapley (1977) all issues are equally likely. However, in some cases as in the EU, the preferences of the European Commission serve as a type of Agenda Setter. Inspired by a work by Owen (1972), Passarelli and Barr (2007) developed a probabilistic measure of power in which the policy positions of the voters are crucial in determining coalition formation. Furthermore, it overcomes the two drawbacks raised above. They applied this measure to the Council of Ministers of the EU, which computes a probabilistic value in a single dimensional policy case. In another paper by Barr and Passarelli (2009), a two dimensional policy space is considered by using principal component analysis. The framework considered therein also allows to understand and interpret Shapley's power index which is a particular case. However, in order to achieve this, the authors use the Shapley scheme and unfortunately inherit the two limitations of spacial indices raised above and pointed out afresh by Benati and Vittucci Marzetti (2013). In addressing these shortcomings, Benati and Vittucci Marzetti (2013) applied a random utility model to the dynamics of coalition formation. To achieve this, the total utility of any player i is modeled as the sum of a deterministic component capturing the predictable utility player i gets from the issue, and a random component modeling the idiosyncratic and unpredictable behavior of the player. A probabilistic generalized spatial value is therefore obtained, which is also a particular class of probabilistic values. Clearly if the random component is zero, then this index reduces to Barr and Passarelli (2009). As in Owen (1971) and Shapley (1977), in the present work, we assume that issues are equally likely and there is no mention of agenda setter as in the three papers above. Our work is not an attempt to solve the two drawbacks raised by Passarelli and Barr (2007) and reconsidered by Benati and Vittucci Marzetti (2013). However, these two papers could be reconsidered using our scheme in which the predictable utility player i gets from an issue depends on the distance between his bliss point and that issue.

The remainder of this paper is organized as follows. Section 2 introduces some basic concepts including voting games, Shapley–Shubik power index in voting games and spatial games. In Section 3, we recall the Owen and Shapley power indices in spatial games and we formally state that Shapley power index is an extension of Owen power index. Section 4 is devoted to the generalization of the distanced-based approach by Owen and the link between this generalization and Shapley power index. Section 5 concludes the study. The proofs are collected in Appendix to ensure clarity and readability.

2. Preliminaries

To start with, we present the basic elements, in particular the definitions of voting games, power indices and their extension to a spatial context. Let n be a nonzero integer and $N = \{1, \ldots, n\}$ be a set of n individuals (voters), 2^N is the set of subsets of N and $|S|$ the cardinality of any subset S of N.

Definition 1. A voting game on N is any pair (N, W) where W, the set of winning coalitions, is a subset of 2^N satisfying: (1): $\varnothing \notin W$, $N \in W$ and (2): $\forall S, T \in 2^N, (S \in W \text{ and } S \subseteq T) \Longrightarrow T \in W$.