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Quantifying risks with exact analytical solutions of derivative pricing distribution

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HIGHLIGHTS

- Path integral approach to the statistical distributions of option pricing.
- Analytical expressions of statistical distribution of bond and bond option pricing.
- Statistical fluctuations and fatty tails of bond and bond option pricing.

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ABSTRACT

Derivative (i.e. option) pricing is essential for modern financial instrumentations. Despite of the previous efforts, the exact analytical forms of the derivative pricing distributions are still challenging to obtain. In this study, we established a quantitative framework using path integrals to obtain the exact analytical solutions of the statistical distribution for bond and bond option pricing for the Vasicek model. We discuss the importance of statistical fluctuations away from the expected option pricing characterized by the distribution tail and their associations to value at risk (VaR). The framework established here is general and can be applied to other financial derivatives for quantifying the underlying statistical distributions.

1. Introduction

Quantitative finance started from the study of stochastic processes. The foundation for understanding the stochastic processes was laid down in the last century thorough Brownian motion [1] by Bachelier, Einstein and Wiener [2–4]. In 1951, Ito established the stochastic calculus for Brownian motion [5]. The stock price fluctuations in time on the financial market can be described by a stochastic process.

Later in 1950s, Samuelson laid down the foundation of option pricing theories via expectations by quantifying both macro- and microeconomics through mathematics [6]. In 1973, Black, Scholes and Merton derived the Black–Scholes partial differential equations for option pricing [7,8].

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In 1977, Boyle related the pricing of options to the simulations of random asset paths, assessing the important roles of Monte Carlo simulations in finance [9]. More recently, in 1994, Derman and Kani [10], Dupire [11] and Rubinstein [12] considered the stochastic nature of the volatility and generalized the Black–Scholes equation in this situation.

In 1977, Vasicek investigated the pricing interest rate products such as bonds. He modeled a short-term interest rate as a random walk and found that the interest rate derivatives can be valued using equations similar to the Black–Scholes equation [13].

In 1986, Ho and Lee found a way of improving Vasicek model with the idea of yield curve fitting or calibration [14]. In 1992, Heath, Jarrow and Morton modeled the random motion of the whole yield curve and generalized Ho and Lee's approach [15].

In addition to the expected pricing discussed, quantifying the statistical fluctuations of equity and bond products as well as their derivatives is critical for addressing the associated risks. Basel committee on banking supervision published on international convergence of capital measurement and capital standards in 2006 [16]. Therefore, banks are now required to assess the risk quantitatively. Moreover, as one can see clearly from the recent financial crisis started from Wall Street and propagated through the whole world, the quantitative risk assessment is crucial for financial stability. Progresses have been made on the equity (stock) pricing models. Furthermore, analytical models of option pricing for both equity and credit so far have often been focused on the expected pricing. The distributions of option pricing revealing high order statistical fluctuations critical for risk assessments, are still challenging to obtain analytically [17–23] despite of numerical efforts such as Monte Carlo approach [9].

In this study, we established a quantitative framework using path integrals to obtain an exact analytical formula for the statistical distribution of the bond and bond option pricing for the Vasicek model. We discuss the importance of statistical fluctuations away from the expected option pricing characterized by the tail of the distribution and the associations to value at risk (VaR) critical in the practice of risk control. Quantifying statistical distributions is also very important in characterizing complex physical and biological systems as well as single molecules dynamics [24–26].

2. Model

In order to explore the statistical distribution of the option pricing, it is important to start with a suitable method. In this study, we use a path integral method [24–33]. This method has its advantages that the expected pricing and associated distribution can all be quantitatively addressed in a relatively straight forward way.

We start with a bond model. Bond is a contract, paid for up-front, that yields a known amount on a known date in the future, the maturity date T. Bonds maybe issued by either governments or companies. The main purpose of a bond issue is the raising of capital, and the up-front premium can be thought of as a loan to the government or the company. The issue related to valuing bond pricing lies on how much one should pay to get a guaranteed amount in certain times. In regarding to the bond options, one aims to find fair values of the contract.

In this section, we will discuss the path integral formulation and the expected pricing of bond and bond options. In the results and discussion section, we will concentrate on the distribution of the derivative pricing of bond and bond options.

2.1. The Vasicek model of bond with fluctuating interest rate

The interest rate for bonds is not a constant and stochastically fluctuates. The Vasicek model for bonds [13] is usually studied from a stochastic differential equation for the short term interest rate r [13,34,35]:

$$\frac{dr}{dt} = a(b - r(t)) + \sigma f(t). \tag{1}$$

The rate dynamics (change) is controlled by two terms: the drift term (first term) and the stochastic term (second term). The parameters a and b model the mean reversion, σ is the volatility of the short term interest rate. a, b and σ are constants in time. The whole dynamics is parameterized in terms of the continuous time variable t. Looking at the drift term, the interest rate will tend to relax (decrease) towards the mean for large r and move up on average for small r. The "force" $f(t)$ represents the stochastic fluctuations:

$$\langle f(t) f(t) \rangle = \delta(t - t), \quad \langle f(t) \rangle = 0. \tag{2}$$

The brackets $\langle\rangle$ indicate a mean value with respect to a Gaussian distribution of zero mean and variance of one.

The interest rate term structure models (including Vasicek model) are for pricing risk-free government bonds. For corporate bonds, additional credit spread and credit risk model are needed.

2.2. The functional for the short term interest rate

From Gaussian noise properties of the stochastic force, we can write down the distribution of "force" f as:

$$p(\{f(s)\}) = e^{-\frac{1}{2}\int ds f^2(s)}. \tag{3}$$