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Asymptotic Green's functions for time-fractional diffusion equation and their application for anomalous diffusion problem

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HIGHLIGHTS

- Asymptotic Green's functions for time-fractional diffusion equation are derived.
- Time-value for the corresponding short and long-time subdivision is defined.
- Initial-value problem for the long-time asymptotic solution is resolved.

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ABSTRACT

Asymptotic Green's functions for short and long times for time-fractional diffusion equation, derived by simple and heuristic method, are provided in case if fractional derivative is presented in Caputo sense. The applicability of the asymptotic Green's functions for solving the anomalous diffusion problem on a semi-infinite rod is demonstrated. The initial value problem for longtime solution of the time-fractional diffusion equation by Green's function approach is resolved.

1. Introduction

Diffusive transport kinetics is usually described by the fundamental diffusion equation based on the second Fick's law, which, in fact, is a mathematical interpretation of mass conservation law. If diffusion process exhibits memory effect [1], which is typical for fractal structure [2], Fick's law is no more applicable because mean square displacement of diffusing species is not always linear with respect to time [3]. Description of this unusual diffusion may be represented by replacement of the corresponding equation with derivative of non-integer order in the range between zero and two. The corresponding equation is called time-fractional diffusion equation. One-dimensional time-fractional diffusion equation is given as follows [4]:

$$\frac{\partial^\alpha C}{\partial t^\alpha} = K \cdot \frac{\partial^2 C}{\partial x^2}, \tag{1}$$

where C is the linear concentration of diffusing species, mole/cm; K denotes fractional diffusion coefficient in porous media, cm^2/s^α; t is time, s; x is coordinate, cm; α is temporal fractional order. Fractional derivative is used in Caputo sense [5],

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because only for Caputo derivative holds mass conservation principle [6]:

$$\frac{\partial^\alpha C}{\partial t^\alpha} = D^\alpha C\left(x, t\right) = \frac{1}{\Gamma\left(m-\alpha\right)} \cdot \int_0^t \left(t-\tau\right)^{m-\alpha-1} \cdot \frac{\partial^m C}{\partial \tau^m} d\tau \tag{2}$$

$m = 1$ if $0 < \alpha < 1$ and $m = 2$ if $1 < \alpha < 2$. For $0 < \alpha < 1$ holds sub-diffusive regime of transport, which is slower comparably to standard Fickian diffusion [7], and for $1 < \alpha < 2$ fast super-diffusion is considered [8]. For $\alpha = 1$ Eq. (1) reduces to standard diffusion equation:

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial^2 C}{\partial x^2}. \tag{3}$$

In the present paper we introduce the asymptotic Greens' functions, based on Fourier–Laplace transforms and Mittag-Leffler function approximations, and apply them to anomalous diffusion problem.

2. Derivation of the asymptotic Green's functions

Applying spatial Fourier and temporal Laplace transform to Eq. (1) gives space-time fractional diffusion equation expressed as follows [9,10]:

$$C\left(k, s\right) = \frac{s^{\alpha-1}}{s^\alpha - K \cdot \left(-i \cdot k\right)^2}. \tag{4}$$

Inverse Laplace transform of Eq. (4) leads to [10]:

$$C\left(k, t\right) = E_\alpha\left(K \cdot \left(-i \cdot k\right)^2 \cdot t^\alpha\right) \tag{5}$$

where E_α denotes special case of one-parameter Mittag-Leffler function [11]:

$$E_\alpha\left(z\right) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\alpha \cdot n + 1\right)}. \tag{6}$$

Mittag-Leffler function is approximated for small and large values of its argument according to the following expression [3,12]:

$$E_\alpha\left(K \cdot \left(-i \cdot k\right)^2 \cdot t^\alpha\right) = \begin{cases} \exp\left[\dfrac{K \cdot \left(-i \cdot k\right)^2 \cdot t^\alpha}{\Gamma\left(m+\alpha\right)}\right], & t < t_m^* \\ \dfrac{1}{K \cdot \left(-i \cdot k\right)^2 \cdot t^\alpha \cdot \Gamma\left(m-\alpha\right)}, & t > t_m^* \end{cases} \tag{7}$$

where t_m^* is given by:

$$t_m^* = \left(-K \cdot \left(-i \cdot k\right)^2\right)^{\frac{1}{\alpha}}. \tag{7a}$$

Applying inverse Fourier transform to Eq. (7) gives asymptotic Green's functions for short and long times respectively:

$$G_\alpha^0\left(x, t\right) = \frac{\exp\left[-\frac{x^2 \cdot \Gamma\left(m+\alpha\right)}{4 \cdot K \cdot t^\alpha}\right]}{2 \cdot \sqrt{\frac{\pi \cdot K \cdot t^\alpha}{\Gamma\left(m+\alpha\right)}}}, \quad t > t^* \tag{8}$$

$$G_\alpha^\infty\left(x, t\right) = \sqrt{\frac{2}{\pi}} \cdot \frac{x}{K \cdot t^\alpha \cdot \Gamma\left(m-\alpha\right)}, \quad t < t^*. \tag{8a}$$

Here the upper index 0 corresponds to Green's function for short time, and the upper index ∞ corresponds to Green's function for long time. In Eqs. (8)–(8a) t^* is given by:

$$t^* = \left|-K^{\frac{1}{\alpha}} \cdot \sqrt{\frac{2}{\pi}} \cdot |x|^{-\frac{2+\alpha}{\alpha}} \cdot \Gamma\left(\frac{2+\alpha}{\alpha}\right) \cdot \sin\left(\frac{\pi}{\alpha}\right)\right|. \tag{9}$$

Eqs. (8)–(9) are valid for $t > 0$, and $0 < \alpha < 2$. For $\alpha = 1$ Eqs. (8)–(8a) reduce to Green's function for standard diffusion equation, because $E_1(z) = \exp[z]$ [11].