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Material line fluctuations slaved to bulk correlations in two-dimensional turbulence

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ABSTRACT

An analogy is pointed out between a polymer chain fluctuating in a two-dimensional nematic background and a freely floating material line buffeted by a two-dimensional turbulent fluid in the inertial (Kraichnan) regime. Under certain conditions, the back-reaction of the line on the turbulent flow may be neglected. The fractal exponent related to the size–contour relation of the material line is connected to a "nematic" correlation function in the bulk.

Theories of turbulence generally focus on the properties of correlation functions rather than make an attempt to solve the Navier–Stokes equation as such [1–3]. If we restrict ourselves to two dimensions [2D], it is the inertial regimes that are important at asymptotically high Reynolds numbers, at scales larger than the Kraichnan dissipation length λ_k [4,5]. These regimes have been studied experimentally in soap films flowing under gravity in set-ups that allow for continuous operations [6]. An interesting correlation function was measured a decade ago [7]. A thin column of water was injected in a turbulent soap film: this could be viewed as a material line being deformed by the 2D turbulence. Amarouchene and Kellay succeeded in measuring the configurational statistics of the evolving fluctuating line [7]. Here, I attempt to connect the line correlation function to that of the bulk turbulence albeit under conditions without symmetry breaking.

The problem is reminiscent of a polymer chain being deformed by a nematic matrix in two dimensions [8,9]. Let us recall the argumentation used to connect the polymer correlation function with the underlying nematic correlations in the limit of strong coupling. The latter are expressed in terms of the director $\vec{n}(\vec{r}) \equiv \exp[i\theta(\vec{r})]$ where the angle $\theta(\vec{r})$ is defined in the 2D complex plane as a function of $\vec{r}$

$$\langle \vec{n}(\vec{r}) \cdot \vec{n}(\vec{r}\,')\rangle_n = \langle e^{i\theta(\vec{r})} e^{-i\theta(\vec{r}\,')}\rangle_n \backsim \left|\vec{r} - \vec{r}\,'\right|^{-\eta}. \quad (1)$$

Here, the average is that defined in thermodynamic equilibrium and the orientational order decays algebraically [9], as is well known. The 2D wormlike chain embedded in the nematic is defined by $z(s) = x(s) + iy(s)$ where s is a point on the contour from one end ($0 \le s \le N$). Because of the strong coupling, the chain is slaved to the nematic. The effective Hamiltonian $\mathcal{H}$ is a functional of z and $\vec{n}$ and consists of the bending energy of the chain, the free energy of the fluctuating nematic and a term signifying the strong coupling of the chain to the nematic as discussed by Nelson et al. [8,9] (for a qualitative treatment of the enslavement, see the Appendix). We have

$$\frac{dz}{ds} = e^{i\theta(s)} \quad (2)$$

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where the right-hand-side is conveniently regarded as a functional of $z(s)$. We therefore obtain

$$\langle\, |z(s) - z(s')|^2\, \rangle_{nc} = \int_0^N \mathrm{d}s \int_0^N \mathrm{d}s' \left\langle \left| \mathrm{e}^{i\theta[z(s)]} \mathrm{e}^{-i\theta[z(s')]} \right| \right\rangle_{nc}. \tag{3}$$

The index nc denotes that two averages have been employed within a canonical ensemble, that is including a factor $\exp(-\mathcal{H}/k_\mathrm{B}T)$ where k_B is Boltzmann's constant and T is the temperature. One average is a functional integration over fluctuations in the nematic (n), the other is a functional integration over chain configurations (c). Inserting Eq. (1) into Eq. (3), we end up with an integral equation. Upon setting $z(N) \backsim N^\nu$, we conclude that [8,9]

$$\nu = \frac{2}{2+\eta}. \tag{4}$$

It is remarkable that this expression is derived without having to use the probability $\exp(-\mathcal{H}/k_\mathrm{B}T)$ itself [8,9]. The problem is whether an expression akin to Eq. (4) is also valid for a material line in a 2D turbulent field.

A chain of length N and mass m_c immersed in a 2D Navier–Stokes fluid behaves like an unattached, one-dimensional flag acting on a fluid area of typical size N^2. One relevant dimensionless parameter $R_1 = m_c/N^2\rho_f$ with ρ_f the fluid density occurs in the theory of a singly attached flag flapping in an Euler fluid [10]. A second parameter R_2 may be viewed as a non-dimensionalized bending energy. The bending energy U_b is of order BN/R_c^2 where R_c is the typical radius of curvature R_c of the bent flag and B is the bending force constant. The bending energy is at most B/N so that the elastic energy density scales as B/N^3. We have to compare this with the fluid Reynolds stress $\rho_f U^2$ where U is a typical velocity of the flag with respect to some background at the far field. We therefore have $R_2 = B/\rho_f U^2 N^3$. We wish to consider the limit where the back reaction of the flag on the fluid is negligible. Von Karman vortices arising at the two ends (when the flag is free) have little effect when $R_2 \gg R_1$ [10,11]. On the other hand, an energy criterion $R_2 \ll 1$ has been introduced by de Gennes [12] to ascertain when passive advection is valid in three dimensions. This criterion has also been applied to 2D turbulent flows [13]. A regime with both $R_2 \gg R_1$ and $R_2 \ll 1$ is easily realizable according to Fig. 3 of Ref. [10].

Of course, a Navier–Stokes fluid is definitely not an Euler fluid even as the kinematic viscosity goes to zero [14] but let us focus on the enstrophy cascade at very high Reynolds numbers. The turbulence is stationary and homogeneous. The rate of dissipation at scales smaller than the injection scale is $\chi = d\langle\omega^2\rangle_h/dt$ where $\langle\rangle_h$ represents an average over an ensemble of realizations of the vorticity $\omega(\vec{r}, t)$ [5]. The inertial regime is here between λ_k and the injection scale; it is scaleless. A material line swaying in the fluid has a viscous boundary layer of size λ_k along its length. At a distance l from this line, the largest eddy must be of order l (at least if the radius of curvature R_c is not too small). But the typical time scale of all the eddies including those in the turbulent boundary layer must be $\chi^{-1/3}$. If we suppose a power law for the material line holds again: $R \sim N^{\nu_h}$, passive advection implies full enslavement of the material line to the flow in the enstrophy cascade regime. I again stress that nowhere in the above analysis of the nematic problem leading to Eq. (4) is explicit use made of a probability function within a canonical ensemble. Hence, one may apply the identical argumentation to a 2D turbulent stationary state with an unknown probability function pertaining to that state. Thus, we simply follow the above line of reasoning to write

$$\nu_h = \frac{2}{2+\eta_h} \tag{5}$$

where the exponent is defined in terms of the hydrodynamic velocity $\vec{v}(\vec{r}) \equiv v(\vec{r})\vec{n}(\vec{r})$ which defines a "polar" director $\vec{n}(\vec{r})$

$$\langle\, \vec{n}(\vec{r}) \cdot \vec{n}(\vec{r}\,')\, \rangle_h \backsim \left| \vec{r} - \vec{r}\,' \right|^{-\eta_h}. \tag{6}$$

The amplitude of the velocity vector is $v(\vec{r})$ and the index h denotes an average over an ensemble of stationary states.

The orientational correlation function given by Eq. (6) appears to have never been computed; in principle, it may hold on general grounds in two dimensions since the Kraichnan regime is scaleless. In the experiments by Amarouchene and Kelly [7], the soap film flows on average in the y direction under gravity. The fluctuation $h(y)$ of the injected material line consisting of pure water is measured in the direction perpendicular to the y axis. Thus, it is expedient to focus on the correlation or structure functions $\langle|\delta h(r)|^n\rangle_h$ with $\delta h(r) \equiv h(y+r) - h(y)$. For $n = 2$, this function scales empirically as r^{ξ_n} where the exponent ξ_n is close to 2 at low rates of flow where the coherent vortices appearing in the 2D fluid are ordered. At higher rates of flow, the 2D film becomes turbulent and the coherent vortices are scattered throughout the turbulent background in a disordered manner. The exponent ξ_2 ultimately reaches a value of about 1.5 continuously until anomalies start to occur related to the integrity of the material line. In Fig. 1 of Ref. [7], the line seems to be attracted to coherent vortices here and there.

Although this issue was not investigated, it is probably safe to posit that the line fluctuations are isotropic implying $\xi_2 \equiv 2\nu_h$. Accordingly, ν_h would range from unity at low rates of flow to about 3/4 at high rates. The relation between the exponent ν_h and η_h given by Eq. (5) can be tested purely empirically as suggested by Hamid Kellay (private communication). In Ref. [7], the exponent ξ_2 is a function of the Reynolds number but this may not mean much; the turbulence in the inertial Kraichnan regime could be fully developed whereas the coherent vortices and their distribution could well still depend on the viscosity of the soap film. Another potential problem in the scaling analysis is that the dimensionless coefficients R_1 and R_2 may need to be renormalized if a power law for the chain size $R(N)$ is posited. Nevertheless, the interaction between