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The unitary elasticity property in a monocentric city with negative exponential population density

Weihua Zhao[1]

Department of Economics, College of Business, University of Louisville, Room 153, 40292, Louisville, KY, United States

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ABSTRACT

The central prediction of the monocentric city model is that real estate prices, density of development, population density, and land rent decline with the distance from urban center. The spatial pattern follows a negative exponential under the conditions derived by Brueckner (1982) and McDonald and Kim (1987). Research has demonstrated that these exponential density gradients have flattened over time due to rising income and falling transportation costs. This paper identifies a heretofore unknown property in a monocentric city with negative exponential population density, the "unitary elasticity property (UEP)." If a city is characterized by a constant density gradient, even if the slope of that gradient is changing over time, the sum of the elasticity of central density and the elasticity of land area with respect to population change will be approximately equal to unity. When this new prediction is tested, it fits US cities fairly well. Further analysis demonstrates that topographic barriers and age of housing stock are the key factors explaining the deviation from the UEP.

1. Introduction

Following Thunen's model of agricultural land use and Alonso's model of urban land use, the monocentric city model of Muth (1969) and Mills (1972) has become the dominant model used to understand urban spatial structure. Under a number of strong assumptions, it can be demonstrated that population and structure density, land and housing prices follow a simple negative exponential. Brueckner (1982) demonstrated that there are three sufficient conditions for the simple negative exponential density functions to hold. First, the housing production is Cobb-Douglas. Secondly, commuting cost is linear in distance. Thirdly, the income-compensated price elasticity of housing demand is unity. These are all stated as assumptions in Muth (1969). Kim and McDonal (1987) further demonstrated the sufficient conditions in Brueckner (1982) imply a zero income elasticity of housing demand when income and price elasticities are constant.

In spite of the facts that the assumptions imposed on the model are strong and may not hold in practice, the predictions that population density, structural density, land values, housing prices and floor area ratio (FAR) fall exponentially with the distance from the city center have been repeatedly tested and confirmed empirically (Clark, 1951; Coulson, 1991; Mills, 1970, 1980; Mills and Tan, 1980; McMillen, 1990, 2006). The repeated confirmation of the predictions has led to a widespread consensus on the validity of the property that the spatial distribution of population can be approximated by a negative exponential density function in most cities.

Based on this negative exponential population density, this paper derives a new property. Specifically, if a city is characterized by a constant density gradient, even if the slope of that gradient is falling over time, the sum of the elasticity of central density and the elasticity of land area with respect to population change will be close to unity. This unitary elasticity property (UEP) prediction has the advantage of being easy to implement using commonly available data. The test reduces to a straightforward hypothesis concerning the sum of two parameters.

The empirical results are generally consistent with the UEP. The sum of the elasticities of central density and land area with respect to population change is generally slightly above unity. Furthermore, an analysis of possible variables explaining departures from unity by individual cities indicates that topographical barriers to development and age of housing stock are the two key factors associated with the deviation from the UEP prediction. Such failure of the UEP in the presence of topographic barriers to development and age of housing stock is not surprising and is actually consistent with the monocentric city model.

E-mail address: weihua.zhao@louisville.edu.
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2. Theoretical framework and predictions

The monocentric city model contains the central features of the models developed by Alonso (1964); Muth (1969) and Mills (1967,1972). Cities are assumed to be monocentric and built on a featureless plain with uniform radial transportation. All employment in one city locates at a center region called the Central Business District (CBD) and all households live outside CBD. Workers in each household have to commute from home to CBD at a constant transportation cost per mile.

Over the years, the key features of the model have been demonstrated using a variety of approaches and assumptions regarding production technology and preferences. Following Brueckner (1987), the approach is presented here as a point of reference to set up the proof of unitary elasticity property.

Housing is measured in units of housing services that are produced by a perfectly competitive industry using land and structure as inputs exhibiting constant returns to scale. The problems associated with the durability depreciation and maintenance of structure inputs are ignored because of the static nature of the model. The housing production function is defined as

$$Q(k) = Q(S(k),\ L(k)) \tag{1}$$

Where $Q(k)$ is the total production of housing services k miles from city center, $S(k)$ and $L(k)$ are structure and land inputs respectively.

The housing producer's objective is to maximize profit.

$$Max p(k)Q(k) - r(k)L(k) - P_s S(k) \tag{2}$$

where $p(k)$ is the rental rate of housing services at distance k, $r(k)$ is rental price of land at distance k, P_s is price of structure input which is constant over distance given the assumption that structure input market is national.

The monocentric city model is built around a household utility maximization problem. The wage paid in the CBD is assumed to be exogenous and identical for each worker. All households have same preference over housing and a composite non-housing good. A household decides how much housing to consume and where to live through utility maximization.

The utility maximization problem can be rewritten as

$$\max_{c,q} U(c,\ q) s.\ t\ \ c+pq=y-tk \tag{3}$$

where c represents non-housing consumption, p represents housing rental price, q is housing consumption, y is income and t represents constant marginal transportation cost. Households' preferences are assumed to be homothetic.

Locational equilibrium implies that all households will reach the same level of utility so that they are indifferent across locations under the assumption that each household can change residential location costlessly. It requires Eq. (4) be satisfied.

$$\frac{dp(k)}{dk} = -\frac{t}{q(k)} \tag{4}$$

Eq. (4) implies that as the household moves further from CBD, the saving in housing price should be sufficient to compensate the increased transportation cost.

Overall equilibrium of the city will be achieved when both land market and labor market are in equilibrium. Equilibrium in land market requires that the price of land at the edge of the city must be equal to the agricultural reservation price of land. Therefore, the distance from city center to the edge of urban area is R, where

$$r(R) = \bar{r} \tag{5}$$

When labor market is in equilibrium, city population must fit in the urbanized land area in the city. The total urbanized land area available for housing is given by $\theta\pi R^2$, where $0 \le \theta \le 1$. This insertion of θ as a constant fraction of land available for housing is the simplest way in which the model can accommodate mixed land uses or the irregular topography in the real world. However, the assumption that θ is constant means that any interruption of development due to topography is uniformly distributed along each radius. Land use for housing in each distance k is exactly equal to the amount of available land, $L(k) = 2\pi\theta k$. Therefore, total population in the city is:

$$N = \int_0^R 2\pi\theta k D(k) dk \tag{6}$$

where $D(k)$ is the population density k miles away from city center.

The negative exponential population density function can be written as

$$D(k) = D_0 e^{-\lambda k} \tag{7}$$

where $D(k)$ represents population density at distance k from CBD, D_0 is the central density at CBD, and λ is the population density gradient.

Substituting Eq. (7) into (6), the equilibrium Eq. (6) can be rewritten as

$$N = D_0 \int_0^R 2\pi\theta k e^{-\lambda k} dk \tag{8}$$

The Unitary Elasticity Property (UEP): In a city characterized by the monocentric city model, the sum of the elasticity of land area with respect to population and the elasticity of central density with respect to population is approximately equal to 1.

$$\frac{\partial lnD_0}{\partial lnN} + \frac{\partial ln\pi\theta R^2}{\partial lnN} \approx 1 \tag{9}$$

The following is a brief outline of the proof of the UEP and details of the proof are provided in Appendix A.

From equilibrium condition, which is city population must fit in the urbanized area

$$N = D_0 \int_0^R 2\pi\theta k e^{-\lambda k} dk \tag{10}$$

the Zero Elasticity Property of the Density Gradient (ZEP) can be derived.

$$\frac{\partial lnD_0}{\partial ln\lambda} + \frac{\partial ln\theta\pi R^2}{\partial ln\lambda} \approx 0 \tag{11}$$

Eq. (11) states that as the density gradient changes, the percentage change in central density will offset the percentage change in land area.

Totally differentiating equilibrium condition (10) with respect to N and λ and rearranging terms gives

$$1 = g(\lambda,\ R)\frac{D_0}{N}\frac{\partial lnD_0}{\partial lnN} + g(\lambda,\ R)\frac{D_0}{\lambda}\frac{\partial lnD_0}{\partial ln\lambda} + D_0{*}g_\lambda(\lambda,\ R) + D_0{*}g_R(\lambda,\ R)\frac{R}{\lambda}\frac{\partial lnR}{\partial ln\lambda} + D_0{*}g_R(\lambda,\ R)\frac{R}{N}\frac{\partial lnR}{\partial lnN} \tag{12}$$

where $g(\lambda, R) = 2\pi\theta\int_0^R ke^{-\lambda k}dk$.

Substituting $N = D_0{*}g(\lambda, R)$ into Eq. (12) and rearranging terms, it can be rewritten as

$$1 = \frac{\partial lnD_0}{\partial lnN} + \frac{R{*}g_R(\lambda,\ R)}{g(\lambda,\ R)}\frac{\partial lnR}{\partial lnN} + g(\lambda,\ R)\frac{D_0}{\lambda}\left[\frac{\partial lnD_0}{\partial ln\lambda} + \frac{\lambda{*}g_\lambda(\lambda,\ R)}{g(\lambda,\ R)} + \frac{R{*}g_R(\lambda,\ R)}{g(\lambda,\ R)}\frac{\partial lnR}{\partial ln\lambda}\right] \tag{13}$$

Under second order Taylor series expansion of $e^{\lambda R}$ about $\lambda = 0$, the following two expressions hold

$$\frac{R{*}g_R(\lambda,\ R)}{g(\lambda,\ R)} = \frac{\lambda^2R^2}{e^{\lambda R}-1-\lambda R} \approx 2 \tag{14}$$

$$\frac{\lambda{*}g_\lambda(\lambda,\ R)}{g(\lambda,\ R)} = \frac{\lambda^2R^2}{1+\lambda R + \frac{(\lambda R)^2}{2!} - 1 - \lambda R} - 2 \approx 0 \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (13) and rearranging terms,