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Semi-analytical analysis for piezoelectric plate using the scaled boundary finite-element method

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ABSTRACT

This paper presents an accurate semi-analytical technique to analyse piezoelectric plates. The proposed technique is built upon the scaled boundary finite-element method. No kinematic and electrostatic assumptions are introduced in the derivation process. The in-plane dimensions are divided into 2D elements and the 3D geometry is obtained by translating the mesh in thickness direction. The through-thickness solutions are expressed analytically with a matrix exponential function. The 3D-onistent nature allows the proposed technique to describe the through-thickness behaviour of piezoelectric plates accurately. Numerical shear locking does not arise. Four numerical examples are presented to highlight the performance of the proposed technique.

1. Introduction

Piezoelectric materials are widely used in sensing devices and actuators of engineering applications due to their unique electro-mechanical coupling characteristics. The advantages of high displacement resolution, fast response, substantial durability and wide frequency band further increase their usages broadly [1,2]. The coupling nature and the integration into composite structures, however, inevitably scale up the difficulty in numerical analysis for material modelling, designs and structural responses [3,4]. Their increasing usage in smart structures and structural health monitoring [5,6], both continuously ensure structural safety, has emphasised the significance in reliably simulating the responses of piezoelectric materials even in their design stage.

Analytical approaches are available to analyse certain types of piezoelectric structures [7–10], albeit they are mostly derived for specific geometries and/or loading conditions. This limits their applications for modern engineering designs that require high degree of flexibility. The numerical analysis of piezoelectric structures, including the optimisation of the design [11,12], is generally conducted using the finite element analysis (FEA) [13–15]. A detailed review was given by [16] on the early FEA development for piezoelectric materials.

The development of plate/shell elements in FEA is the main research focus for piezoelectric materials in the past two decades [17–19]. This is mainly due to their popular plate-form applications, such as the bending actuators and sensors [6,2,20,21]. However, different with pure elastic materials, plate assumptions do not necessarily apply to piezoelectric plates. High order through-thickness behaviour appears even for small thickness-to-length ratio [22]. This leads to the need of using higher order theories [23–25], equivalent single layer formulations [17,26] or layerwise formulations [27,4] for the element development. The combination of single layer (for deformation) and layerwise (for electric potential) formulations was also found in the literature for developing piezoelectric elements [28,29]. For a summary of the advantages and disadvantages of the aforementioned approaches, readers are referred to two past reports [26,18].

In order to deal with the numerical locking phenomenon, the above approaches require conventional remedies such as the reduced integration technique [19], assumed strain field [30], mixed formulation [4] and the mixed interpolation of tensorial components approach [17,26]. Instead of using plate/shell elements, Braess [31] presented a 3D-finite element formulation for thin piezoelectric structures with the use of reduced integration method. This assures the consistency of 3D constitutive behaviour of piezoelectric materials. There exist a great amount of other numerical methods to effectively analysing piezoelectric plates. Meshless method is one of those being popularly developed in the last two decades and it leads to a number of successful methods to simulate piezoelectric structures [32–35]. Recently Nguyen-Van et al. [36] extended the strain smoothing method, which is based on a meshfree conforming nodal integration [37], to develop a smoothed four-node piezoelectric element. Sladek et al. [38] also developed a meshless method based on the local Petrov–Galerkin approach to analyse piezoelectric plates with functionally graded material properties.

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The authors have recently developed a unified 3D consistent technique to solve plate bending problems [39,40]. This technique is derived solely from the three-dimensional theory. The in-plane dimension of a plate is divided into 2D finite elements. The solution is expressed analytically in the through-thickness direction. The 3D-consistent nature allows this technique to be applicable for both thin and thick plates. It has been proven the developed technique does not suffer from transverse shear locking [39,40]. No *ad hoc* factors, such as the shear correction factor, are introduced. This technique is based on the scaled boundary finite-element method [41], which solves boundary value problems semi-analytically. The scaled boundary finite-element method has been successfully applied to model stress singularities occurring at crack tips, bi-material interfaces and multi-material wedges [42–45]. Its first application of analysing stress singularities in piezoelectric composites is developed by Mayland and Becker [46]. Comparable studies are subsequently carried out by the authors [47,48]. Its applications are also found in solving problems in transient elastodynamics and in unbounded domains [49–53]. Owing to its advantages in accurately evaluating singular stress fields and in flexibly meshing, the scaled boundary finite-element method is employed for simulating crack propagation [54–56]. Recently, this method is extended to plate and laminated composite analysis in the framework of Kirchhoff's plate theory [57–59].

In this paper, the technique in [39,40] is extended to analyse thin to moderately thick piezoelectric plates by including the electromechanical coupling in the developed formulations. The general solutions, i.e., the deflections and the electric potential, are expressed analytically as an exponential matrix function of the thickness coordinate. The stiffness matrix is constructed by treating the piezoelectric plate as a stack-up of two identical layers. This allows the proposed technique to give the solutions of the top, middle and bottom planes of the piezoelectric plate. The solutions of the three planes are subsequently used to compute the coefficients of the quadratic through-thickness solutions. A scheme to tailor the 3D stiffness matrix for the applications of piezoelectric bending sensor or extending actuator is also developed. Owing to the analytical expression of the solutions, numerical shear locking issue does not arise. Moreover, with the use of high order spectral elements, the proposed technique is able to accurately model piezoelectric plates with curved boundaries without sacrificing computational efficiency.

The paper is organised as follows. Section 2 demonstrates the extension of the scaled boundary finite-element method for piezoelectric plates. The proposed solution technique is subsequently presented in Section 3. It is followed by the numerical studies in Section 4 and finally, a conclusion is given in Section 5.

2. Scaled boundary finite-element method for piezoelectric plates

This section formulates the scaled boundary finite-element method for piezoelectric plates. Only the key equations related to this development for piezoelectric materials are included. The readers are referred to [39,40] on plates of elastic materials for further details.

2.1. 3D governing equations of piezoelectric plates

A piezoelectric plate of constant thickness t is shown in Fig. 1. The z coordinate of the Cartesian coordinate system is chosen along the transverse (through-thickness) direction of the plate. The x, y coordinates are parallel to the midplane. As the piezoelectric plate is handled as a 3D structure, the displacement components along x-, y- and z-directions are denoted as $u_x = u_x(x,y,z)$, $u_y = u_y(x,y,z)$ and $w = w(x,y,z)$, respectively. Similarly, the electric potential is denoted as $\phi = \phi(x,y,z)$. Following the original formulations for elastic plates [39], the displacement and electric potential vector $\{\bar{u}\} = \{\bar{u}(x,y,z)\}$ is arranged as $\{\bar{u}\} = [w, u_x, u_y, \phi]^T$. The strains and electric fields $\{\bar{\varepsilon}\} = \{\bar{\varepsilon}(x,y,z)\}$ are expressed as

$$\{\bar{\varepsilon}\} = [\varepsilon_z, \varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}, -E_z, -E_x, -E_y]^T = [L]\{\bar{u}\} \tag{1}$$

with the differential operator

$$[L] = \begin{bmatrix} \frac{\partial}{\partial z} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & 0 & \frac{\partial}{\partial y} \end{bmatrix}. \tag{2}$$

The stresses and electric displacements $\{\bar{\sigma}\} = \{\bar{\sigma}(x,y,z)\}$ follow from linear piezoelectricity

$$\{\bar{\sigma}\} = [\sigma_z, \sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{xz}, D_z, D_x, D_y]^T = [H]\{\bar{\varepsilon}\} \tag{3}$$

with the constitutive matrix $[H]$ given as

$$[H] = \begin{bmatrix} [C] & [e]^T \\ [e] & -[\epsilon] \end{bmatrix}, \tag{4}$$

in which $[C]$ contains the elastic constants, $[e]$ contains the piezoelectric constants and $[\epsilon]$ contains the permittivity. The equation of equilibrium with vanishing body force and body charge is written as

$$[L]^T\{\bar{\sigma}\} = 0. \tag{5}$$

The bottom and top surfaces of the plate may be subjected to surface traction and electric charge $\{\bar{q}\} = [f_z, f_x, f_y, q]^T$. Various supporting conditions can be applied on the sides of the plate.

The in-plane dimensions are discretised into 2D spectral elements [60]. A typical 3rd order spectral element is shown in Fig. 1 with 4 nodes across each in-plane direction. The geometry of an element is represented by interpolating its nodal coordinates $\{x\}$ and $\{y\}$ using the shape functions $[N] = [N(\eta,\zeta)] = [N_1(\eta,\zeta), N_2(\eta,\zeta), \ldots]$ formulated in the local coordinates η and ζ:

$$x(\eta,\zeta) = [N]\{x\} \tag{6a}$$

$$y(\eta,\zeta) = [N]\{y\} \tag{6b}$$

The geometry of the 3D plate is obtained by translating the 2D mesh along the z-direction. This corresponds to placing the scaling centre of the scaled boundary finite-element method at infinity [49,61].

The governing differential equations (Eqs. (1)–(5)) are formulated in the scaled boundary coordinates z, η and ζ. Only the coordinate transformation between the x, y coordinates and the η, ζ coordinates are required. Based on Eq. (6b), the transformation is expressed as

$$\begin{Bmatrix} \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \tag{7}$$

with the Jacobian matrix

$$[J] = \begin{bmatrix} x_{,\eta} & y_{,\eta} \\ x_{,\zeta} & y_{,\zeta} \end{bmatrix}. \tag{8}$$

Inverting the transformation in Eq. (7) leads to

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{,\zeta} & -y_{,\eta} \\ -x_{,\zeta} & x_{,\eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} \tag{9}$$