Spatial Statistics 18 (2016) 72–85

Contents lists available at ScienceDirect

Spatial Statistics

journal homepage: www.elsevier.com/locate/spasta

A spectral mean for random closed curves

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ARTICLE INFO

Article history:
Received 10 July 2015
Accepted 3 February 2016
Available online 15 February 2016

In memory of J. Harrison

Keywords:
Image analysis
Missing data
Random closed set
Spectral analysis

ABSTRACT

We propose a spectral mean for closed sets described by sample points on their boundaries subject to mis-alignment and noise. We derive maximum likelihood estimators for the model and noise parameters in the Fourier domain. We estimate the unknown mean boundary curve by back-transformation and derive the distribution of the integrated squared error. Mis-alignment is dealt with by means of a shifted parametric diffeomorphism. The method is illustrated on simulated data and applied to photographs of Lake Tana taken by astronauts during a Shuttle mission.

1. Introduction

Many geographical or biological objects are observed in image form. The boundaries of such objects are seldom crisp, for example due to measurement error and discretisation, or because the boundaries themselves are intrinsically indeterminate (Burrough and Frank, 1996). Moreover, the objects may not be static in the sense that if multiple images are taken, the objects may have been deformed.

One attempt to model natural objects under uncertainty is fuzzy set theory (Zimmermann, 2001). However, the underlying axioms are too poor to handle topological properties of the shapes to be modelled and cannot deal with correlation. Similarly, the belief functions that lie at the heart of the Dempster–Shafer theory (Dempster, 1967; Shafer, 1976) do not necessarily correspond to the containment function of a well-defined random closed set (Molchanov, 2005).

Proper stochastic geometric models for natural objects are scarce. Indeed, the Boolean model (Molchanov, 1997), defined as the union of independent copies of a simple geometric object, to this date dominates the random set literature. It owes its popularity to the fact that it is analytically

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http://dx.doi.org/10.1016/j.spasta.2016.02.002
2211-6753/

tractable, but its realisations do not match the irregular shapes observed in natural phenomena. The modification in which objects may be subject to transformation is known as a deformable template model (Grenander and Miller, 2007).

Apart from the dearth of models, another complication is that, since the family of closed sets in the plane is not linear, it is not straightforward to define what is meant by a mean set. Indeed, many different suggestions have been made, often with a specific context in mind. For instance, the radius vector mean is taylor-made for star-shaped objects, the Vorob'ev expectation for fuzzy sets. This context dependence has obvious disadvantages. More specifically, the radius vector mean does not apply to sets that are not star shaped, the Vorob'ev expectation is a function of the coverage function only so that it cannot take into account spatial coherence and fine detail. The Fréchet expectation (Fréchet, 1948) on the other hand can be defined quite generally as a distance minimiser in a metric space and – in contrast to the previously mentioned definitions – is extendable to higher orders, but the computational burden may be large. For further details and examples, see Molchanov (2005).

In this paper, we extend an interesting approach suggested by Jónsdóttir and Jensen (2005) for star-shaped objects deformed by noise to objects that are not necessarily star-shaped by modelling their boundary as a closed curve. Additionally, we propose a spectral mean for such random sets and carry out inference in the Fourier domain. The approach is easy to implement and, moreover, the integrated squared error can be computed in closed form.

A complication is that the curves need to be well-aligned in the sense that their parametrisations must be in correspondence. Similar problems arise in shape registration (Dryden and Mardia, 1998), deformable template matching (Grenander and Miller, 2007), signal extraction (Bigot and Gendre, 2013) or image warping (Glasbey and Mardia, 2001). Although the terminology varies, the objective is always the same, namely to find a transformation between objects so that they resemble one another. The context determines the class of transformations and the criterion for evaluating the resemblance. Examples of the former include rigid motions or time shifts; resemblance can be based on Euclidean distance between landmarks (Besl and McKay, 1992), Hausdorff or image distances (Baddeley and Molchanov, 1998), or on intrinsic properties of the object (e.g. arc length and curvature) (Sebastian et al., 2003) or the group of transformations (Kendall, 1984). Further discussion and examples can be found in Davies et al. (2008).

Since we work on a space of smooth cyclic functions, we shall use the (expected) L_2 distance on this function space to measure the quality of alignment and we follow Younes (2010) in reparametrising a curve by diffeomorphisms. Note that, in contrast to function estimation in one dimension, a reparametrisation does not affect the appearance of the curve.

This paper is organised as follows. We begin by recalling basic facts about planar curves, cyclic Gaussian random processes and spectral analysis. Then we formulate a model for sampling noisy curves, carry out inference in the Fourier domain and quantify the error. We present a simulation study to assess performance. Finally, we describe how to deal with mis-alignment between curves and illustrate the approach on images of a lake in Ethiopia.

2. Noisy curves

In this section we recall basic facts about planar curves, Fourier bases and cyclic Gaussian random processes.

2.1. Planar curves

Throughout this paper, we model the boundary of a random object of interest by a smooth closed curve.

Consider the class of functions $\Gamma : I \to \mathbb{R}^2$ from some interval I to the plane. Define an equivalence relation $\sim$ on the function class as follows: Two functions Γ and Γ' are equivalent, $\Gamma \sim \Gamma'$, if there exists a strictly increasing function φ from I onto another interval I' such that $\Gamma = \Gamma' \circ \varphi$. Note that φ is a homeomorphism. The relation defines a family of equivalence classes, each of which is called a curve. Its member functions are called parametrisations. Since the images of two parametrisations of