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Sub-regions without subdomain partition with boundary elements

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ABSTRACT

This paper presents a simple and effective numerical procedure to model domains with sectorial heterogeneous properties using the boundary element method. The physical problem is divided into a complete homogenous domain and other non homogeneous sectors. Matrices similar to the standard H boundary element matrix are constructed for each sector, which are related to the energy stored in them. The assembly of the coefficients related to the sectorial matrices in the final boundary element system is done through the direct introduction of unknown variables at internal auxiliary points. Comparatively to the sub-regions technique, the proposed procedure is advantageous, since the effect of interface approximations is attenuated and computer implementation of its corresponding numerical model is simpler.

1. Introduction

One of the most unsuitable applications to the Boundary Element Method (BEM) concerns the modeling of non-homogeneous medium problems, very common cases in soil mechanics. Considering that the heterogeneity is located sectorally, the use of sub-regions is still the most efficient approach [1]. As the procedure is based on a simple concept considering domain partition, meaningful changes were not observed concerning it along the time [2,3]. However, due the insertion of internal boundaries, in certain complex situations, the sub-regions technique is unsatisfactory because it becomes costly and inadequate for programming.

Unfortunately, there are no other efficient BEM approach regarding the solution of this important class of problems; thus domain methods such as the Finite Element Method [4,5] and the Finite Difference Method [6], are usually chosen to model it.

In other important engineering branches such as fracture mechanics, in which the BEM is more efficient than other traditional discrete methods, similar restrictions can occur. If layered-materials are involved, sub-regions technique must be introduced and the size of the final matrix can be increased significantly. To preserve the advantages of the BEM in this case, occasionally some strategies are proposed to reduce the size of the final matrix when a large number of elements need to be used [7]. Anyway, the creation of additional internal boundaries and consequent disturbance in the approximation still persists.

This work presents an alternative to treat this category of problems that, due to its simplicity, can accredit the BEM to further elaborate applications without implementation difficulties, highlighting interesting problems such as plasticity and time dependent cases that have distinct properties on the domain.

With the proposed technique, once the complete domain is constituted by sectorial homogeneities, it is modeled superposing a homogeneous complete domain and other sub-domains with different properties. All sectors are linked through influence coefficients, which in the BEM procedure are generated by integrations performed in non homogeneous boundaries, with the source points located in the complete or surrounding domain. Particularly, energy principles support the proposed procedure, which makes it even more distinct to the sub-region concept.

In a way the idea of this work is approximately presented in models that consider localized domain actions, which consist of another kind of problem for which the BEM requires additional auxiliary techniques to become efficient. Loeffler and Mansur [8] used this approach to account for sectorial loads with the Dual Reciprocity technique [9]. Also the solution of problems of interaction between soil and structure employing the FEM and the BEM together [10], as well as cases of integration between zoned plate bending [11,12] and plate-beam-column integrated systems [13,14] employ partially the idea shown in this article, in which common nodal points of different systems are assembled in a general matrix. It is worth mentioning that Wagdy and Rashed [15] also propose a formulation where they deduce and assemble

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an additional stiffness matrix to boundary integral equation. Indeed, the authors connect different sub-domains (such as plates with columns in structures) using internal points. The final procedure and objectives of them is distinct from the approach presented here since it was not proposed a mathematical model to deal with sub-domains having different properties. They are highlighted in bibliography of the present paper because the methodology presented here uses as Wagdy and Rashed [15] the idea of linkage of distinct domains using internal points. As additional contribution in the present paper, theoretical considerations are presented to justify a model where the energy of each sub-domain is computed such as is done with a domain source in Poisson's problems.

Although the procedure is general, for convenience, in this work only stationary scalar problems governed by Laplace Equation are approached.

2. Bem approach for Laplace equation

Consider a two-dimensional domain that represents a thermal or mechanical field at steady state, which has homogeneous and isotropic properties and there are no sources, sinks or external domain actions that contribute directly to the field. Taking the basic variable u(**X**) as a scalar potential, the differential equation associated with this problem is given by the Laplace equation, that is:

$$K\nabla^2 u(\mathbf{X}) = 0 \tag{1}$$

In Eq. (1), ∇(.) is the Nabla operator. The meaning of the physical property represented by K depends of the physical problem considered and when homogeneity and isotropy are assumed, it can obviously be omitted when considering Laplace's problems. Considering finite domains, the essential and natural boundary conditions are prescribed on closed boundaries $\Gamma(\mathbf{X})$, given respectively by the following equations:

$$u(\mathbf{X}) = \bar{u}(\mathbf{X}), \quad on \quad \Gamma_u(\mathbf{X}); \quad and \quad u_{,i}\ n_i(\mathbf{X}) = \bar{q}(\mathbf{X}), \quad on \quad \Gamma_q(\mathbf{X}) \tag{2}$$

In Eq. (2) n_i is the unit outward normal vector on the boundary $\Gamma(\mathbf{X})$, that is composed by the union of boundary regions $\Gamma_u(\mathbf{X})$ and $\Gamma_q(\mathbf{X})$.

As is well known in the literature, the classic BEM formulation can be deduced through a weighted residual expression [16] or through an integral equation in which the fundamental solution u*(ξ;**X**) is used as an auxiliary function over the domain Ω(**X**) [17], so providing the following equation for Laplace's problems:

$$K\int_{\Omega} \nabla^2 u(\mathbf{X}) u^*(\xi;\mathbf{X}) d\Omega(\mathbf{X}) = 0 \tag{3}$$

It can be verified that Eq. (3) represents the heat energy balance in the physical system, which in this case is in equilibrium, indicating the equivalence between the conservative diffusive energy accumulated and the work of external actions applied to it. This becomes evident through application of mathematical procedures typical of BEM, including the discretization process, that generated matrices **H** and **G** as indicated below:

$$K[\mathbf{H}]\{\mathbf{u}\} = K[\mathbf{G}]\{\mathbf{q}\} \tag{4}$$

While in the Finite Element Method the resulting matrix expressions represent the Newtonian equilibrium, although they have been deducted from energy minimization principles [18], regarding to BEM the left hand side of Eq. (4) represents clearly the internal work of resistive actions while the right hand side is the work of external actions. The mean of energy balance of the BEM is used later on in this work.

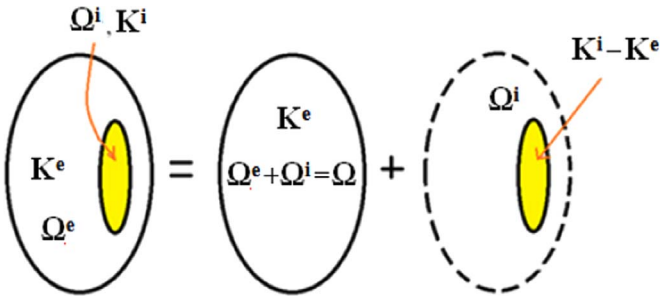

Fig. 1. Complete and sectorial domains with homogeneous properties.

3. Proposed heterogeneous model

For instance, consider a domain composed by two regions with different physical properties, as shown in Fig. 1.

In Fig. 1, the complete domain Ω(X) is composed by the sum of Ω^e and Ω^i; both K^e and K^i are physical properties, constant inside each sub-domain. In this formulation, unlike what is done in the traditional sub-regions approach, a complete or surrounding domain is elected, such is shown above, with homogeneous properties, e.g., K^e.

Consider that the kernel of the integrals is comprised by integrable functions and the properties are constant inside each sub-domain. Supposing then $K^i = K^e + K^*$ the following equations can be written:

$$\begin{aligned}
&\int_{\Omega} K(X) u(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega(\mathbf{X}) \\
&= K^e \int_{\Omega^e} u^e(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega^e(\mathbf{X}) + K^i \int_{\Omega^i} u^i(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega^i(\mathbf{X}) \\
&= K^e \int_{\Omega^e} u^e(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega^e(\mathbf{X}) + \int_{\Omega^i} (K^e + K^*) u^i(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega^i(\mathbf{X}) \\
&= K^e \int_{\Omega} u^e(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega^e(\mathbf{X}) + K^* \int_{\Omega^i} u^i(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega^i(\mathbf{X}) = 0
\end{aligned} \tag{5}$$

Now, the property K^e comprises the complete domain. Eliminating K^* it results:

$$K^e \int_{\Omega} u(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega(\mathbf{X}) = (K^e - K^i) \int_{\Omega^i} u^i(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega^i(\mathbf{X}) \tag{6}$$

Eq. (6) could have been proposed and interpreted by energy principles. As it stands, there is equilibrium of diffusive energy and work of fluxes on both sides of equality. In the proposed method it is necessary only to quantify the diffusive energy present in the sub-domain, such as is done in the accounting of a source or any external action.

In this sense, to clarify ideas, consider that instead of a sector heterogeneity there is a body action located on a domain $\Omega^i(\mathbf{X})$. Then, an integral form similar to left hand side of Eq. (6) can be established for the Poisson's problem, that is:

$$K^e \int_{\Omega} u(X)_{,ii}\ u^*(\xi;\mathbf{X}) d\Omega(\mathbf{X}) = \int_{\Omega^i} p^i(\mathbf{X}) u^*(\xi;\mathbf{X}) d\Omega^i(\mathbf{X}) \tag{7}$$

This problem is sketched in Fig. 2.

Well known mathematical treatments with the BEM can be performed to left hand side of the Eq. (7), that is, application of integration by parts twice to the integral and use of properties of the Dirac Delta function to eliminate the domain integral. A boundary integral equation concerning a Poisson's problem follows, given by:

$$\begin{aligned}
&c(\xi)u(\xi) + \int_{\Gamma} u(\mathbf{X}) q^*(\xi;\mathbf{X}) d\Gamma(X) - \int_{\Gamma} q(\mathbf{X}) u^*(\xi;\mathbf{X}) d\Gamma(X) \\
&= -\frac{1}{K^e} \int_{\Omega^i} p^i(\mathbf{X}) u^*(\xi;\mathbf{X}) d\Omega^i(X)
\end{aligned} \tag{8}$$

The numerical computation of the right hand side of Eq. (8) can be carried out by many BEM schemes, that is: integration by cells