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Shape reconstructions from phaseless data

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ABSTRACT

For the inverse scattering problem from a sound-soft obstacle with full far field data, a simple hybrid method was proposed by Lee [9]. In this research, we consider the case where as data only the modulus of the far field are available. The aim is to reconstruct the shape of the obstacle with this limited data.

1. Introduction

The inverse problems whose aims are to discover or to reconstruct the geometry of an inaccessible unknown object from the measurement of the scattered field at some convenient place have attracted more and more attention to a great extent due to their diverse applicability for example in non-destructive testing, seismic exploration or medical imaging. One of the most studied inverse problem is the inverse obstacle scattering problem which targets at the reconstruction of the shape and location of the unknown scatterer from the knowledge of the scattered field or the corresponding far field pattern. Theoretically, the unknown scatterer can be uniquely determined with full data, i.e., the far field pattern for all incident directions for a fixed wave number [8,4]. However, in the case where the phase information is not available, the unknown object is not identifiable due to the translation invariance of the modulus of the far field pattern (see [6]).

In this article we consider as a model the time-harmonic scattering problem from a sound-soft obstacle. The aim of the corresponding inverse problem is to recover the shape of the unknown scatterer when only the intensity of the far field pattern caused by a planar incident wave is measured.

Since our problem lies in an unbounded domain, it is advantageous to treat this problem by the boundary integral equations method. This kind of method transforms the domain into a compact set in most cases which is beneficial both from the theoretical and the numerical points of view. On the one hand, it allows elegant analysis as far as the unique solvability and the stability of the problem is concerned with the power of functional analysis. On the other hand, it reduces the computational cost by decreasing the dimension. In most cases, the problem itself is converted into a system of boundary integral equations in the framework of Fredholm integral equation of the second kind with compact integral operators. For the details, we refer to [1,5].

The plan of the paper is as follows. For the sake of completeness and also the introduction of notation, in Section 2, a brief summary of the main results of the direct problem including the unique solvability will be presented. In Section 3 we consider an inverse problem and a two-by-two system of integral equations will be derived whose equivalence to the inverse scattering problem will also be established. An iteration scheme for numerical computation of the inverse problem will be proposed in Section 4. This will be followed by some numerical examples in Section 5. Some concluding remarks will be given in the final section.

2. Direct scattering problem

Let $D \subset \mathbb{R}^2$ be a simply connected domain with C^2 boundary ∂D. We denote the closure of D by $\bar{D}$. The direct scattering problem that we are considering, for a given incident plane wave $u^i(x, d) := e^{ik\langle x,d\rangle}$ with a wave number $k > 0$ and a unit vector d giving the direction of propagation, is to find a solution $u^s \in C^2(\mathbb{R}^2\backslash\bar{D}) \cap C(\mathbb{R}^2\backslash D)$ to the Helmholtz equation

$$\Delta u^s + k^2 u^s = 0, \quad \text{in } \mathbb{R}^2\backslash\bar{D} \tag{1}$$

which satisfies the Dirichlet boundary conditions

$$u^s = -u^i \quad \text{on } \partial D \tag{2}$$

and the Sommerfeld radiation condition

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$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u^s}{\partial\nu}-iku^s\right)=0,\quad r:=|x| \tag{3}$$

uniformly for all directions $\hat{x}:=\frac{x}{|x|}$. Note here that the boundary condition (2) is equivalent to the homogeneous condition $u=0$ for the total field $u=u^i+u^s$. This is why we call the obstacle sound-soft.

Using boundary integral equations, this direct problem can be solved via the boundary layer approach, see for example the monograph [1]. Indeed, in terms of the fundamental solution to the Helmholtz equation in $\mathbb{R}^2$

$$\Phi(x,y):=\frac{i}{4}H_0^{(1)}(k|x-y|),\quad x\neq y,$$

one can define the solution ansatz

$$u^s(x)=\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)ds(y)+i\int_{\partial D}\Phi(x,y)\varphi(y)ds(y)\quad x\in\mathbb{R}^2\backslash\bar{D} \tag{4}$$

by a combination of a double layer potential and a single layer potential (see [1]). For the unique solvability of the direct scattering problem, the following theorem can be addressed.

Theorem 1. *The direct scattering problem has a unique solution given by (4), where the density function $\varphi\in C(\partial D)$ is the (unique) solution to the following boundary integral equation*

$$\varphi(x)+2\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)ds(y)+2i\int_{\partial D}\Phi(x,y)\varphi(y)ds(y)=-2u^i(x) \tag{5}$$

for $x\in\partial D$.

At this place we want to point out that in the scattering problem, one is particularly interested in the far field pattern or the scattering amplitude u_∞ of the scattered field u^s. The far field pattern describes the behavior of the scattered wave at the infinity and its relation to the scattered field u^s is given by

$$u^s(x)=\frac{e^{ik|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x})+O\left(\frac{1}{|x|}\right)\right\}\quad |x|\to\infty$$

uniformly for all directions $\hat{x}\in\Omega:=\{x\in\mathbb{R}^2 \mid |x|=1\}$. The one-to-one correspondence between radiating waves and their far field patterns is established by Rellich's lemma. Using the asymptotics of the Hankel functions, the far field pattern of the scattered field to our direct problem is

$$u_\infty(\hat{x})=\lambda_1\int_{\partial D}\langle\nu(y),\hat{x}\rangle e^{-ik\langle\hat{x},y\rangle}\varphi(y)ds(y)+i\lambda_2\int_{\partial D}e^{-ik\langle\hat{x},y\rangle}\varphi(y)ds(y) \tag{6}$$

with the constants $\lambda_1=\frac{1-i}{4}\sqrt{\frac{k}{\pi}}$, $\lambda_2=\frac{1+i}{4\sqrt{k\pi}}$ and the density function φ given by Theorem 1.

The (direct) scattering problem can thus be summarized as the process of computing the far field pattern of the scattered wave from a given obstacle which is mathematically formulated as the consecutive solving of (5) and (6). For further discussions, we rewrite them in the following operator form:

$$\begin{cases}B(\partial D,\varphi)=-2u^i\\ F(\partial D,\varphi)=u_\infty\end{cases} \tag{7}$$

The boundary operator B describes the boundary condition which is solved for the density function φ for a given obstacle with one incident planar wave. The far field operator F calculates the far field pattern u_∞ of the scattered field generated by the obstacle.

3. Inverse problem

After solving direct problem for a sound-soft scatterer and introducing the notation in the last section, we are now in a position to discuss the inverse obstacle scattering problem.

The associated inverse problem we would have considered would be the reconstruction of the obstacle from the modulus of the far field pattern $|u_\infty(\cdot,d)|$, i.e., the determination of the shape and the location of the obstacle. This is however not possible since the modulus of the far field pattern $|u_\infty|$ is invariant under translation as shown in [6]. More precisely, for any vector $s\in\mathbb{R}^2$, the far field pattern $u_{\infty,s}$ from the shifted obstacle $D_s:=\{x+s \mid x\in D\}$ can be described by the relation

$$u_{\infty,s}(\hat{x})=e^{iks\cdot(d-\hat{x})}u_\infty(\hat{x}),\quad \hat{x}\in\Omega \tag{8}$$

Therefore, it does not make sense to consider the inverse problem of identification of the unknown scatterer in the present setting. Instead, we consider the sole determination of the shape of the scatterer from the modulus of the far field pattern $|u_\infty(\cdot,d)|$ for a single incident direction d at a fixed frequency k.

To solve this inverse problem, it is equivalent to solve the following equation

$$|F(\partial D,\varphi)|=|u_\infty| \tag{9}$$

for the unknown boundary ∂D. Apparently, this is a nonlinear equation with two unknowns. For the linearization of it, Newton's method is favorite because of its simplicity. For this purpose, we need to take the Fréchet derivatives of the right-hand side w.r.t. these two variables. It is hence beneficial to treat the equivalent equation

$$\overline{F(\partial D,\varphi)}F(\partial D,\varphi)=\overline{u_\infty}u_\infty \tag{10}$$

Instead of processing the inverse problem promptly with (9) or (10), let us first take one more look at its origin, i.e., the system (7). This system describes the full solution procedure for the direct scattering problem. The inverse problem, as this nomenclature verbally suggests, should be solved via the same system with the givens and the unknowns interchanged. From this point of view, we begin the treatment of our inverse problem with the following system:

$$\begin{cases}B(\partial D,\varphi)=-2u^i\\ \overline{F(\partial D,\varphi)}F(\partial D,\varphi)=\overline{u_\infty}u_\infty\end{cases} \tag{11}$$

Formally, this is just the system (7) with its second equation replaced by (10) to fit our inverse problem. It can be shown as in [7] that the solving of this system amounts to the solving of our inverse problem.

Given the modulus of the far field pattern $|u_\infty|$, this system is to be solved for the two quantities ∂D, φ. Apparently, both equations are nonlinear in ∂D. The second equation is also nonlinear in φ. To work with Newton's method, linearizations w.r.t. both variables for both operators are needed. This means that Fréchet derivatives w.r.t. both variables for both operators must be calculated. Besides nonlinearity, all the Fréchet derivatives from the above are compact operators. Hence simultaneous regularizations for ∂D and φ are needed. This is the basic idea of the nonlinear integral equations method [7,3,2]. The solving of the system is, however, much involved.

At this place we note that the variable φ is somehow redundant. Indeed, our problem is to find the boundary ∂D only, it is in our solution method that the variable φ comes up. From this point of view, less effort on φ is desirable.

From the solution theory of the direct problem, we know that the first equation of the above system is well-posed. According to Theorem 1, the density function φ is uniquely determined when the boundary ∂D is given. Once this φ is computed in the framework of the forward problem, there is only one unknown left in the second equation of system (11). This means that only one linearization and one regularization are needed.