



A boundary element formulation for the heat equation with dissipative and heat generation terms



Roberto Pettres^{a,b,c,*}, Luiz Alkimin de Lacerda^{a,1}, José Antonio Marques Carrer^b

^a Institute of Technology for Development LACTEC, Department of Civil Structures DPEC Curitiba, CEP: 81531-090, P. O. Box: 19067, Paraná, Brasil

^b UFPR – Federal University of Paraná, Postgraduate Program in Numerical Methods in Engineering, Curitiba, Paraná, Brasil

^c UFPR – Federal University of Paraná – Advanced Campus of Jandaia do Sul, Jandaia do Sul, Paraná, Brasil

ARTICLE INFO

Article history:

Received 27 March 2014

Received in revised form

7 November 2014

Accepted 9 November 2014

Available online 5 December 2014

Keywords:

Boundary element method

Transient heat equation

Time independent fundamental solution

ABSTRACT

This article presents a formulation of the Boundary Element Method (BEM) for the study of heat diffusion in isotropic and homogeneous media. The proposed formulation has a time independent fundamental solution obtained from the two-dimensional Laplace equation. Consequently, the formulation is called D-BEM since it has domain integrals in the basic integral equation. The first order time derivative that appears in the integral equations is approximated by a backward finite difference scheme. Internal dissipative and heat generation terms are considered in the analyses. The results from the numerical model are compared with the available analytical solutions. The correlation estimator R^2 is employed to validate the numerical model and to demonstrate the accuracy of the proposed formulation.

© 2014 Elsevier Ltd. All rights reserved.

1. Introduction

This work is concerned with the development of a Boundary Element Method (BEM) formulation for the solution of the heat equation with the presence of dissipative and heat generation terms. Before presenting the proposed formulation, a brief discussion concerning the different formulations that can be used for the solution of time-dependent problems by the BEM must be carried out. Different BEM formulations arise according to the nature of the fundamental solution employed, that is, according to the use of time-dependent or time independent fundamental solutions. In the first case, the so-called TD-BEM formulations arise (TD means time-domain); see, for instance, Wrobel [1], Young et al. [2]. Here, only the boundary discretization is required for the solution of problems with null initial condition, whereas problems that present non-homogeneous initial condition are solved with the discretization of the part of the domain where it appears. On the other hand, the steady-state fundamental solution, much simpler than the time-dependent one, can be used for performing time-domain analyses. The counterpart of the simplicity of the fundamental solution is the presence of a domain integral, whose kernel is constituted by the product of the fundamental solution by the

first order time derivative of the temperature, in the basic BEM integral equation. The transformation of this domain integral into boundary integrals, by means of suitable interpolation functions, generates the DR-BEM formulation, DR meaning dual reciprocity; see Tanaka et al. [3], Singh and Tanaka [4], Ochiai et al. [5,6]. If the domain integral is kept into the integral equation, one has the so-called D-BEM formulation, with D meaning domain. In the D-BEM formulation, the discretization of the entire domain is mandatory. The disadvantage of the domain discretization is counterbalanced by the simplicity of the formulation and reliable results it produces; see Taigbenu and Liggett [7] and Carrer et al. [8]. Both the DR-BEM and the D-BEM formulations require the adoption of a time-marching scheme, that is, an approximation for the first-order time derivative: the simpler choice usually falls on the backward finite difference scheme [9].

Although this scheme can always produce accurate results, other alternatives were sought recently: in Carrer et al. [8], the backward finite difference is combined with the Houbolt approximation [10] and also an approach based on the subdomain collocation method, Finlayson [11], is presented. The level of accuracy, however, was the same achieved by the backward difference scheme. It is important to mention that attention has also been devoted to meshless approaches; see for instance Boztosun and Charafi [12].

The context in which the present work is situated is based on a D-BEM formulation and employs the backward finite difference as an approximation for the first order time derivative of the temperature.

* Corresponding author. Tel.: +55 43 9840 2040; fax: +55 41 3361 6111.

E-mail addresses: pettres@ufpr.br (R. Pettres),

alkimin@lactec.org.br (L.A. de Lacerda), carrer@ufpr.br (J.A.M. Carrer).

¹ Tel.: +55 41 3361 6108; fax: +55 41 3361 6111.

The main contribution of the work is the incorporation of the heat generation and of the dissipative terms. The domain discretization employed triangular linear cells, in which it was assumed a constant variation for the time derivative of the temperature. The boundary discretization employed linear elements. It is important to note that extra domain integrals appear in the formulation due to the presence of the dissipative and heat generation terms. As these extra domain integrals contain the fundamental solution in their integrand, their evaluation does not present any additional effort, as the domain integral related to the time derivative of the temperature has been already computed. Three examples are included and the numerical results are compared with the analytical solutions.

2. Mathematical model

The heat equation in a two dimensional isotropic and homogeneous domain Ω with boundary Γ is written as follows

$$\nabla^2 u(X, t) = \frac{1}{\alpha} \frac{\partial u(X, t)}{\partial t}$$

$$X \in \Omega, \quad X = (x, y) \tag{1}$$

where α represents the coefficient of thermal diffusivity measured in m^2/s , u is the temperature, X is the field point and t is the time variable.

The boundary conditions are:
Essential

$$u(X, t) = \hat{u}(X, t) \quad X \in \Gamma_u \tag{2}$$

Natural

$$q(X, t) = \frac{\partial u(X, t)}{\partial n(X)} = \hat{q}(X, t) \quad X \in \Gamma_q \tag{3}$$

The initial condition at $t=t_0$ is given by

$$u(X, t) = u_0(X, t_0) \quad X \in \Omega \tag{4}$$

3. D-BEM formulation

The integral equation of the D-BEM formulation for the heat equation can be written as follows

$$C(\xi)u(\xi, t) = \int_{\Gamma} u_*(\xi, X)q(X, t)d\Gamma - \int_{\Gamma} q_*(\xi, X)u(X, t)d\Gamma$$

$$+ -\frac{1}{\alpha} \int_{\Omega} \frac{\partial u(X, t)}{\partial t} u_*(\xi, X)d\Omega \tag{5}$$

where $C(\xi)$ is a geometric coefficient at the collocation point ξ , q is the thermal flux and u_* and q_* are the fundamental solution and its normal derivative, respectively.

The expression of the fundamental solution $u_*(\xi, X)$ is given by Greenberg [13],

$$u_*(\xi, X) = \frac{1}{2\pi} \ln\left(\frac{1}{r}\right) \tag{6}$$

where $r = |X - \xi|$ is the distance between field, X , and collocation, ξ , points.

The derivative of the fundamental solution with respect to the normal direction to the boundary is given by

$$q_*(\xi, X) = \frac{\partial u_*}{\partial r} \frac{dr}{dn} = -\frac{1}{2\pi r} \frac{dr}{dn} \tag{7}$$

where n is the outward normal to the boundary.

For simplicity, the time derivative presented in Eq. (5) is approximated by the backward finite difference formula [9]:

$$\frac{\partial u(X, t)}{\partial t} = \frac{u(X, t + \Delta t) - u(X, t)}{\Delta t} \tag{8}$$

Replacing (8) in (5) and grouping terms conveniently, one has

$$C(\xi)u(\xi, t + \Delta t) = \int_{\Gamma} u_*(\xi, X)q(X, t + \Delta t)d\Gamma - \int_{\Gamma} q_*(\xi, X)u(X, t + \Delta t)d\Gamma$$

$$+ -\frac{1}{\alpha\Delta t} \left(\int_{\Omega} u(X, t + \Delta t)u_*(\xi, X)d\Omega - \int_{\Omega} u(X, t)u_*(\xi, X)d\Omega \right) \tag{9}$$

Eq. (9) can be used recursively for the solution of the problem, starting at time t_m and determining the variables at time t_{m+1} . According to Wrobel [1], the critical time step, Δt_c , can be estimated as

$$\Delta t_c \leq \frac{L_j^2}{2\alpha} \tag{10}$$

where L_j is the boundary element size.

For the solution of the problem, the boundary Γ is divided into boundary elements Γ_j , approximating the geometry of each element Γ_j with linear interpolation functions. Along each element the variables of the problem (potential and flux) are approximated by linear continuous approximation functions, see Brebbia [14].

In the domain discretization, it was assumed a constant behavior of the potential within each cell. A generic cell is defined by vertices $k_1(x_1, y_1)$, $k_2(x_2, y_2)$ and $k_3(x_3, y_3)$, as seen in Fig. 1.

The cell integrals are calculated numerically using a local coordinate transformation as illustrated in Fig. 2, and the global coordinates are defined by,

$$x = (1-U)x_1 + U[(1-V)x_2 + Vx_3]$$

$$y = (1-U)y_1 + U[(1-V)y_2 + Vy_3] \tag{11}$$

The Jacobian of the transformation (Eq. (12)) is equal to the double of the cell area A :

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix} = 2A \tag{12}$$

The non-singular cell integrals are computed with Gaussian quadrature according to

$$\int_{\Omega} u_*(\xi, X)d\Omega = \sum_{i=1}^M \sum_{j=1}^M \int_0^1 \int_0^{1-U} u_*(\xi_i, X_j) |J_j| dV dU \tag{13}$$

for $i \neq j$

where M is the number of cells.

In the cases where the source point is located in the cell domain, a weak singularity is present and the integration is carried out using the third order coordinate transformation proposed by Telles [15]. For source points in the centroid of the cell, a cell subdivision is performed as illustrated in Fig. 3 and each triangular part is integrated with the same approach.

After applying Eq. (9) to the boundary nodes and internal points, one obtains the following system of equations:

$$\begin{bmatrix} \mathbf{H}^{bb} & \frac{1}{\alpha\Delta t} \mathbf{M}^{bd} \\ \mathbf{H}^{db} & \mathbf{I} + \frac{1}{\alpha\Delta t} \mathbf{M}^{dd} \end{bmatrix} \begin{bmatrix} \mathbf{u}^b \\ \mathbf{u}^d \end{bmatrix}_{m+1} = \begin{bmatrix} \mathbf{G}^{bb} \\ \mathbf{G}^{db} \end{bmatrix} \begin{bmatrix} \mathbf{q}^b \end{bmatrix}_{m+1} + \frac{1}{\alpha\Delta t} \begin{bmatrix} \mathbf{M}^{bd} \\ \mathbf{M}^{dd} \end{bmatrix} \begin{bmatrix} \mathbf{u}^d \end{bmatrix}_m \tag{14}$$

In Eq. (14), \mathbf{H} and \mathbf{G} are matrices which result from the boundary integrals related to $q_*(\xi, X)u(x)$ and to $u_*(\xi, X)q(x)$, respectively; the matrix \mathbf{M} results from the domain integrals and \mathbf{I} is the identity matrix. The first element of each superscript indicates the position of the source point and the second, the position of the field point, with b indicating boundary and d indicating domain. The subscript $m+1$ indicates the time $t_{m+1} = (m+1)\Delta t$ and the subscript m , the time $t_m = m\Delta t$, where Δt

Download English Version:

<https://daneshyari.com/en/article/512209>

Download Persian Version:

<https://daneshyari.com/article/512209>

[Daneshyari.com](https://daneshyari.com)