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A novel linear triangular element of a three-dimensional displacement discontinuity method

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ABSTRACT

Since only the boundary of the domain requires discretization, the boundary element method (BEM) is very efficient for the semi-infinite or infinite rock-related engineering problems, e.g., hydraulic fracturing in reservoir stimulation and rock cutting during excavation. A real fracture in the solid is usually of an arbitrary geometry in three dimensions, which usually requires a three-dimensional displacement discontinuity method (3D DDM) to determine the deformation and stress field in order to achieve reliable results. However, the use of 3D DDM with triangular elements is limited by the singularities of the integral either within or nearby the domain. In this paper, a novel linear triangular element with three nodes on its vertices is proposed. The analytical integral expressions of this linear triangular element are also theoretically derived. A solution procedure is also described which can be applied to determine the displacement and stress field around a three-dimensional fracture inside the infinite solid. The accuracy of these results are compared with the analytical solutions of the displacements and stresses induced by a pressurized penny-shaped. This procedure takes a shorter time and requires less elements than the usual constant DDM when achieving the same accuracy.

1. Introduction

Many important practical problems in science and engineering can be reduced into mathematical models that belong to a class of problems known as boundary value problems [5,6]. These are all characterized by a region of interest R enclosed within a boundary C. If R is a two dimensional region, C will be a bounding contour. If R is a three dimensional region then C is a bounding surface. For a fracture or cavity inside an infinite region, C will be the fracture or cavity surface. Only the boundary C is divided into elements in BEM, which is totally different from the network of elements for the entire region R in a finite element method (Fig. 1).

The displacement discontinuity in BEM can be considered as a relative displacement between the two surfaces of a plane crack [24,2,23]. The 2D DDM of constant displacement element in linear elastic mechanics was systematically developed by Crouch [5] who proposed a general numerical algorithm to solve various boundary value problems. The 3D DDM is based on the analytical solution to the problem of a discontinuous displacement over a finite area in an infinite region. For a crack of arbitrary non-planar shape in an infinite three-dimensional space, the crack is usually required to be divided into triangular elements. Even though the strong singularities in boundary integral equation [7,29,1,21,22], the 3D DDM on the triangular element is improved from constant element to higher order element. Kuriyama et al. [9,10] proposed the 3D DDM with the boundary surface division into constant triangular elements and briefly presented the solution procedure. Davey and Hinduja [8] established the integral solutions for triangular elements applied to potential problems. Based on Davey's idea, Milroy et al. [14] obtained the general solution by a linear combination of the three integral solutions on the nodes of a triangular element. Napier and Malan [16] and Shou et al. [25] described a higher order displacement discontinuity method in conjunction with shape functions. Nikolski et al. [17,18] presented a boundary element by decoupling the complex variables for the shear tractions and real variable equation for the normal traction for solving 3D elastostatic problems of fractured rock under gravity. Then the complex variables BEM is extended to incorporate higher order approximations of the displacement discontinuities Nikolski [19,20,15].

The recent development of multi-stage hydraulic fracturing technology in a horizontal well in petroleum reservoir requires hydraulic fracturing simulators which can consider the three-

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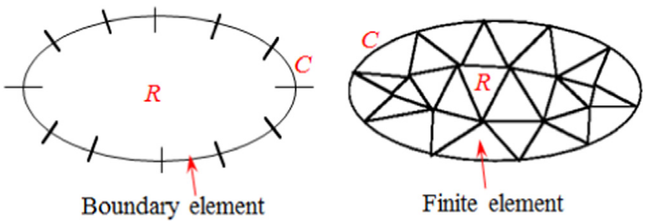

Fig. 1. The element in BEM and FEM.

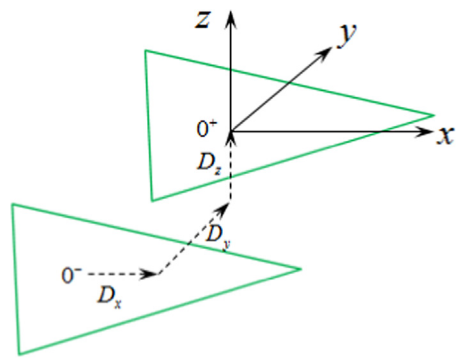

Fig. 2. Triangular element in the 3D DDM.

dimensional formation and stress field around the fracture [3,4]. Physically, any fracture in an infinite three-dimensional space should be in three dimensions. The 3D DDM can be applied to simulate this kind of engineering problem numerically by assuming that the formation is a linear elastic medium. For the strong singularities in boundary integral equation, researchers hope to improve both the computational efficiency and accuracy utilizing the analytical integration instead of the numerical integration. Therefore, a novel linear displacement triangular element with three nodes on its vertices is proposed in this paper. If the displacement discontinuities on these three nodes keep the same, this linear displacement triangular element will be simplified as the constant displacement triangular. The idea of this linear triangular element can also be applied to determine linear triangular element in a fictitious stress method [26]. The analytical integral solutions of this linear triangular element are also theoretically derived. A solution procedure is also briefly described which can be applied to determine the stresses field around a three-dimensional fracture inside the solid. The accuracy of these results is compared with the analytical solutions of displacements and stresses induced by a pressurized penny-shaped crack [27].

2. Three-dimensional displacement discontinuity element

As shown in Fig. 2, the two surfaces of a crack are separated with relative displacements. The two surfaces can be distinguished by the positive side ($z=0_+$) and negative side ($z=0_-$). Crossing from one side to the other, the displacement discontinuities are given by the relative displacement in three orthogonal directions respectively [23,24].

$$\begin{cases} D_x(x,y)=u_x(x,y,0_-)-u_x(x,y,0_+) \\ D_y(x,y)=u_y(x,y,0_-)-u_y(x,y,0_+) \\ D_z(x,y)=u_z(x,y,0_-)-u_z(x,y,0_+) \end{cases} \tag{1}$$

where D_i (i=x,y,z) are the displacement discontinuities in x,y,z direction respectively. u_i (i=x,y,z) are the displacements in x,y,z direction respectively.

To numerically implement the displacement discontinuity element in a boundary element method, we need the analytical solution for the displacement discontinuity over an area which is usually defined as a boundary element. The general form solution for a displacement discontinuity element can be expressed as follows [6,23,24,26]:

$$\begin{cases} u_x=\left[2(1-\nu)\Phi_{x,z}-z\Phi_{x,xx}\right]-z\Phi_{y,xy}-\left[(1-2\nu)\Phi_{z,x}+z\Phi_{z,xz}\right] \\ u_y=-z\Phi_{x,xy}+\left[2(1-\nu)\Phi_{y,z}-z\Phi_{y,yy}\right]-\left[(1-2\nu)\Phi_{z,y}+z\Phi_{z,yz}\right] \\ u_z=\left[(1-2\nu)\Phi_{x,x}-z\Phi_{x,xz}\right]+\left[(1-2\nu)\Phi_{y,y}-z\Phi_{y,yz}\right]+\left[2(1-\nu)\Phi_{z,z}-z\Phi_{z,zz}\right] \end{cases} \tag{2}$$

$$\begin{cases} \sigma_{xx}=2G\left\{\left[2\Phi_{x,xz}-z\Phi_{x,xxx}\right]+\left[2\nu\Phi_{y,yz}-z\Phi_{y,xxy}\right]+\left[\Phi_{z,zz}+(1-2\nu)\Phi_{z,yy}-z\Phi_{z,xxz}\right]\right\} \\ \sigma_{yy}=2G\left\{\left[2\nu\Phi_{x,xz}-z\Phi_{x,xyy}\right]+\left[2\Phi_{y,yz}-z\Phi_{y,yyy}\right]+\left[\Phi_{z,zz}+(1-2\nu)\Phi_{z,xx}-z\Phi_{z,yyz}\right]\right\} \\ \sigma_{zz}=2G\left\{-z\Phi_{x,xzz}-z\Phi_{y,yzz}+\left[\Phi_{z,zz}-z\Phi_{z,zzz}\right]\right\} \\ \sigma_{xy}=2G\left\{\left[(1-\nu)\Phi_{x,yz}-z\Phi_{x,xxy}\right]+\left[(1-\nu)\Phi_{y,xz}-z\Phi_{y,xyy}\right]-\left[(1-2\nu)\Phi_{z,xy}+z\Phi_{z,xyz}\right]\right\} \\ \sigma_{yz}=2G\left\{\left[-\nu\Phi_{x,xy}-z\Phi_{x,xyz}\right]+\left[\Phi_{y,zz}+\nu\Phi_{y,xx}-z\Phi_{y,yyz}\right]-z\Phi_{z,yzz}\right\} \\ \sigma_{xz}=2G\left\{\left[\Phi_{x,zz}+\nu\Phi_{x,yy}-z\Phi_{x,xxz}\right]-\left[\nu\Phi_{y,xy}+z\Phi_{y,xyz}\right]-z\Phi_{z,xzz}\right\} \end{cases} \tag{3}$$

where $\sigma_{ij}, u_i (i,j=x,y,z)$ represent the stress and displacement of Kelvin's problem in three dimensional space respectively. G, ν are the shear modulus and Possion's ratio of the isotropic medium, respectively. $\Phi_{i,j}, \Phi_{i,jk}, \Phi_{i,jkl} (i,j,k,l=x,y,z)$ are the partial derivatives with respect to each corresponding suffix of the following function,Φ_i. The function, Φ_i, from a triangular element against a point can be represented by [12,15]

$$\Phi_i(x,y,z)=\frac{1}{8\pi(1-\nu)}\iint_\Delta \frac{D_i(\xi,\eta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}\mathrm{d}\xi\,\mathrm{d}\eta \quad i=x,y,z \tag{4}$$

where (x,y,z) are the coordinates of an arbitrary point in an infinite 3D space. (ξ, η, 0) are the local coordinates of the loading point. $D_i(\xi,\eta)$ represents the discontinuous displacement on point (ξ, η, 0). Because of the strong singularity of Eq. (4) when (x,y,z) goes to (ξ, η, 0), it is extremely difficult to get the analytical integration [7].

The constant displacement element (Fig. 3a), which means the same displacement over the entire element, will give lower accuracy, compared with the linear displacement element (Fig. 3b). For an opened fracture, the fracture width varies continuously from the crack tip to the crack center. The normal displacement $D_z(\xi,\eta)$ of each triangular element represents the fracture width physically. For constant displacement element, the triangular element only has one node, which is usually the gravity point (Fig. 4a). Hence, $D_i(\xi,\eta)$ is not related to the coordinates (ξ, η, 0) in the element, i.e., constant displacement element, then the Eq. (4) can be expressed by [9,10,26].

$$\Phi_i(x,y,z)=\frac{D_i}{8\pi(1-\nu)}\iint_\Delta \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}\mathrm{d}\xi\,\mathrm{d}\eta \quad i=x,y,z \tag{5}$$

By contrast with constant displacement element, a linear triangular element (Fig. 4b) has three nodes at its vertexes, P_1 (x_1,y_1,0), P_2 (x_2,y_2,0), P_3 (x_3,y_3,0). The displacements in each element form a plane in the three dimensional space, and the equation of this plane reads

$$\begin{vmatrix} \xi-x_1 & \eta-y_1 & D_i(\xi,\eta)-D_i^{(P_1)} \\ x_{21} & y_{21} & D_i^{(P_2)}-D_i^{(P_1)} \\ x_{31} & y_{31} & D_i^{(P_3)}-D_i^{(P_1)} \end{vmatrix}=0 \quad i=x,y,z \tag{6}$$

$D_i(\xi,\eta)$ in Eq. (6) is solved as

$$D_i(\xi,\eta)=\left[1-\frac{(\xi-x_1)(y_{31}-y_{21})+(\eta-y_1)(x_{21}-x_{31})}{x_{21}y_{31}-x_{31}y_{21}}\right]D_i^{(P_1)} + \frac{(\xi-x_1)y_{31}-(\eta-y_1)x_{31}}{x_{21}y_{31}-x_{31}y_{21}}D_i^{(P_2)}+\frac{(\eta-y_1)x_{21}-(\xi-x_1)y_{21}}{x_{21}y_{31}-x_{31}y_{21}}D_i^{(P_3)} \tag{7}$$