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Mathematics and Computers in Simulation 143 (2018) 3–13
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

An exact formula for the L_2 discrepancy of the symmetrized Hammersley point set

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Received 5 September 2015; received in revised form 5 December 2015; accepted 7 December 2015
Available online 8 January 2016

Abstract

The process of symmetrization is often used to construct point sets with low L_p discrepancy. In the current work we apply this method to the shifted Hammersley point set. It is known that for every shift this symmetrized point set achieves an L_p discrepancy of order $\mathcal{O}\left(\sqrt{\log N}/N\right)$ for $p \in [1, \infty)$, which is best possible in the sense of results by Roth, Schmidt and Halász. In this paper we present an exact formula for the L_2 discrepancy of the symmetrized Hammersley point set, which shows in particular that it is independent of the choice for the shift.

Keywords: L_2 discrepancy; Hammersley point set; Davenport's reflection principle

1. Introduction and statement of the result

The local discrepancy $\Delta(\alpha, \beta, \mathcal{P})$ of an N-element point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in the unit square $[0, 1)^2$ is defined as

$$\Delta(\alpha, \beta, \mathcal{P}) = A([0, \alpha) \times [0, \beta), \mathcal{P}) - N\alpha\beta$$

for $\alpha, \beta \in (0, 1]$. In this definition $A([0, \alpha) \times [0, \beta), \mathcal{P})$ is the number of indices $0 \le n \le N - 1$ satisfying $\boldsymbol{x}_n \in [0, \alpha) \times [0, \beta)$. The L_p discrepancy of a point set $\mathcal{P}$ in $[0, 1)^2$ is defined as

$$L_p(\mathcal{P}) = \frac{1}{N}\left(\int_0^1 \int_0^1 |\Delta(\alpha, \beta, \mathcal{P})|^p \, \mathrm{d}\alpha \, \mathrm{d}\beta\right)^{\frac{1}{p}}$$

for $p \in [1, \infty)$. For $p \to \infty$ we obtain the notable star discrepancy. In this work we do not study this kind of discrepancy directly, but it should be mentioned that there is a remarkable asymptotic relation between the L_p discrepancy and the star discrepancy (see [8]). The L_p discrepancy is a quantitative measure for the irregularity of distribution of a point set $\mathcal{P}$ in $[0, 1)^2$, see e.g. [4,11,15]. It is also related to the worst-case integration error of a

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http://dx.doi.org/10.1016/j.matcom.2015.12.002

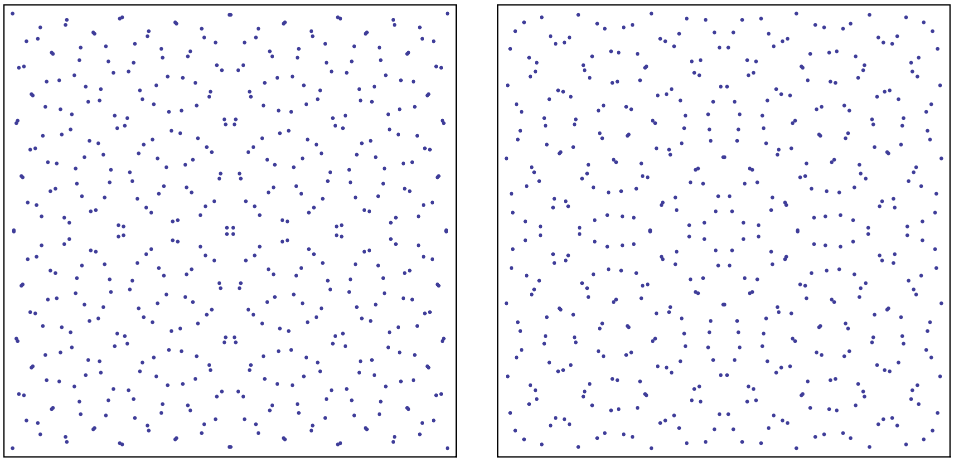

Fig. 1. The symmetrized Hammersley point sets $\mathcal{H}_8^{\mathrm{sym}}(\boldsymbol{\sigma}_i)$ for $i = 1, 2$ where $\boldsymbol{\sigma}_1 = (0, 0, 0, 0, 0, 0, 0, 0)$ and $\boldsymbol{\sigma}_2 = (0, 1, 0, 1, 0, 1, 0, 1)$. The L_2 discrepancy is 0.00255571 ... in both cases.

quasi-Monte Carlo rule, see e.g. [3,14,16]. It is well known that for every $p \in [1, \infty)$ there exists a constant $c_p > 0$ with the following property: for the L_p discrepancy of any point set $\mathcal{P}$ consisting of N points in $[0, 1)^2$ we have

$$L_p(\mathcal{P}) \geq c_p \frac{\sqrt{\log N}}{N}, \tag{1}$$

where log denotes the natural logarithm. This was first shown by Roth [18] for $p = 2$ and hence for all $p \in [2, \infty)$ and later by Schmidt [19] for all $p \in (1, 2)$. The case $p = 1$ was verified by Halász [6].

Here we consider digit shifted Hammersley point sets. Let therefore m be a positive integer and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in \{0, 1\}^m$ a dyadic shift. We define the point set

$$\mathcal{H}_m(\boldsymbol{\sigma}) := \left\{ \left(\frac{t_m}{2} + \frac{t_{m-1}}{2^2} + \cdots + \frac{t_1}{2^m}, \frac{s_1}{2} + \frac{s_2}{2^2} + \cdots + \frac{s_m}{2^m} \right) : t_1, \ldots, t_m \in \{0, 1\} \right\},$$

where $s_j = t_j \oplus \sigma_j$ for all $j \in \{1, \ldots, m\}$ (the operation $\oplus$ denotes addition modulo 2). The point set $\mathcal{H}_m(\boldsymbol{\sigma})$ contains 2^m elements. We obtain the classical Hammersley point set $\mathcal{H}_m$ with 2^m points by choosing $\boldsymbol{\sigma} = (0, 0, \ldots, 0)$. Additionally, we define the m-tuple $\boldsymbol{\sigma}^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_m^*)$ by $\sigma_j^* = \sigma_j \oplus 1$ for all $j \in \{1, \ldots, m\}$. Then we introduce the symmetrized Hammersley point set $\mathcal{H}_m^{\mathrm{sym}}(\boldsymbol{\sigma})$ as

$$\mathcal{H}_m^{\mathrm{sym}}(\boldsymbol{\sigma}) := \mathcal{H}_m(\boldsymbol{\sigma}) \cup \mathcal{H}_m(\boldsymbol{\sigma}^*).$$

This point set has 2^{m+1} elements and can be regarded as symmetrized since $\mathcal{H}_m^{\mathrm{sym}}(\boldsymbol{\sigma})$ may also be written as the union of $\mathcal{H}_m(\boldsymbol{\sigma})$ with the point set

$$\left\{ \left(x, 1 - \frac{1}{2^m} - y \right) : (x, y) \in \mathcal{H}_m(\boldsymbol{\sigma}) \right\}.$$

Fig. 1 shows examples of two symmetrized Hammersley point sets.

The concept of symmetrizing point sets plays an important role in finding point sets with the optimal order of L_p discrepancy in the sense of (1). Davenport [2] used this method in 1956 to construct for the first time a two-dimensional point set with an L_2 discrepancy of order $\mathcal{O}\left(\sqrt{\log N}/N\right)$, and therefore showing that the lower bound (1) is sharp for $p = 2$. For this reason, the symmetrization method we use here is often referred to as Davenport's reflection principle.

It is known that $L_p(\mathcal{H}_m)$ is only of order $\mathcal{O}((\log N)/N)$ for all $p \in [1, \infty)$ (see [17]). However, in [7, Theorem 2] it was shown with tools from harmonic analysis (the Haar function system and the Littlewood–Paley inequality) that the symmetrized Hammersley point set achieves an L_p discrepancy of order $\mathcal{O}\left(\sqrt{\log N}/N\right)$ for all $p \in [1, \infty)$ independently of the shift $\boldsymbol{\sigma}$. This order is best possible in the sense of (1). For the case $p = 2$, this result follows