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A Monte Carlo estimation of the mean residence time in cells surrounded by thin layers

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Highlights

- Presents a probabilistic interpretation of diffusion in medium with a thin layer of low diffusivity.
- Tests the model to compute the mean-residence time in a brain imaging problem.
- Improves the method for estimating an exponential rate from some distribution function estimated by a Monte Carlo method.

Abstract

We present a new Monte Carlo method to estimate the mean-residence time of a diffusive particle in a domain surrounded by a thin layer of low diffusivity. Through a homogenization technique, the layer is identified with a membrane. The simulations use a stochastic process called the snapping out Brownian motion the density of which matches suitable transmission conditions at the membrane. We provide a benchmark test which is a simplified form of a real-life problem coming from brain imaging techniques. We also provide a new algorithm to adaptively estimate the exponential rate of the tail of the distribution function of the probability to be in the domain using Monte Carlo simulations.

Keywords: Semi-permeable membrane; Snapping out Brownian motion; Thin layer problem; Mean-residence time; First eigenvalue estimation

1. Introduction

Understanding the dynamics of a diffusive process in a medium with interfaces is of great importance both for modeling and simulation. An interface could be a permeable or a semi-permeable barrier. It arises as a limit of a sharp change in the properties of the underlying material. In geophysics, the inclusion of rocks, fissures, ..., leads to such interfaces, just to cite one among the many possible domains of applications (see, *e.g.*, [2,13]).

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We consider here a medium with a "thin layer" of low diffusivity seen in the limit as a semi-permeable barrier the particle has difficulty to pass through (see [22, Chap. 13] or [19] for a proof relying on homogenization techniques, and [11, Th. 2] for a probabilistic proof in one dimension). The membrane surrounding living cells is an example of such a layer. A brain imaging technique such as *diffusion Magnetic Resonance Imaging* (dMRI) records the mean square displacement of particles of water [6]. Reconstructing the brain activity is done by estimating the diffusion coefficient through an inverse problem. Understanding the diffusive behavior of particles in living tissue, where thin layers are frequently present, is then essential for applying Monte Carlo methods [17].

In brain imaging, the *mean residence time* (MRT) in the cells is an important macroscopic parameter related to the rate of convergence toward equilibrium. It could serve as an input for simplified, homogenized models such as double porosity models [3,4,16,21]. The MRT is also commonly used in pharmacokinetics and geophysics [15,20,25].

We propose a way to estimate the MRT in media containing thin layers with Monte Carlo simulations using continuous processes instead of random walks as in [21]. We simulate paths of a stochastic process the density of which solves the PDE giving the concentration of the species of interest. To avoid the large computational cost induced by the layers, which forces to use a small time step around it, we replace the process by a simpler one called the *snapping out Brownian motion* [11]. This is equivalent to replacing the thin layer by a semi-permeable membrane. We then show how to estimate the MRT from the Monte Carlo simulation. By the ways, we improve the estimation procedure of a first eigenvalue problem proposed in [12]. Finally, we provide a one-dimensional benchmark test which shows the effectiveness of our approach unless the permeability of the membrane is too low.

2. A Monte Carlo estimation of the mean residence time

The concentration $C(t,x)$ of the diffusive species (water, ...) follows an equation of the type

$$\partial_t C(t,x) = \nabla \cdot (D(x)\nabla C(t,x)),\ t>0,\ x\in\Omega \tag{1}$$

in a box Ω of dimension $d=1,2,3$ with a scalar diffusivity and periodic boundary conditions. Equivalently, we could solve (1) on a periodic medium composed of property translated copies of Ω. We consider here a simplified situation where Ω contains an interior part Ω_{intra} separated from an exterior part Ω_{extra} by a layer Ω_{layer} of constant width $\epsilon>0$. (See Fig. 1.) The surface separating Ω_{layer} and Ω_{intra} (resp. Ω_{extra}) is denoted by Γ_- (resp. Γ_+). We assume that $\Gamma_\pm$ are of class $\mathscr{C}^2$.

For $D_0,\mu>0$, the diffusivity is $D(x)=D_0$ for $x\in\Omega_{\text{intra}}\cup\Omega_{\text{extra}}$, while $D(x)=D_{\text{layer}}=\mu\epsilon$ for $x\in\Omega_{\text{layer}}$.

Under the above conditions, (1) is well posed and has a unique (up to an additive constant), periodic solution for any initial condition which is bounded and measurable.

2.1. The transmission condition

Since D is constant except on the interfaces Γ_- and Γ_+, we recast (1) into a *transmission problem*.

We denote by $n_-(x)$ (resp. $n_+(x)$) the normal derivative at a point $x\in\Gamma_-$ (resp. $x\in\Gamma_+$) which is directed toward Ω_{intra} (resp. Ω_{extra}). The concentration C is of class $\mathscr{C}^1$ on the left and right of $\Gamma_\pm$. Besides, it satisfies

$$\partial_t C(t,x) = D(x)\triangle C(t,x),\ t>0,\ x\notin\Gamma_\pm, \tag{2}$$

with

$$[n_\pm(x)\cdot D(x)\nabla C(t,x)]_{\Gamma_\pm}=0 \text{ and } [C(t,x)]_{\Gamma_\pm}=0, \tag{3}$$

where at a point $x\in\Gamma_\pm$,

$$[f(x)]_{\Gamma_\pm} = \lim_{\delta\to 0} f(x+\delta n_\pm(x)) - f(x-\delta n_\pm(x)).$$

The condition (3) is called the *transmission condition* (see [8, §III.13, p. 224]). It specifies the continuity of the *flow* $n_\pm\cdot D(x)\nabla C(t,x)$ across the interface $\Gamma_\pm$ as well as the continuity of the concentration. The variational formulation of (2)–(3) is given in Appendix A.2.