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On an explicit lower bound for the star discrepancy in three dimensions

Florian Puchhammer

Institute of Financial Mathematics and Applied Number Theory, University Linz, Altenbergerstraße 69, 4040 Linz, Austria

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Abstract

Following a result of D. Bylik and M.T. Lacey from 2008 it is known that there exists an absolute constant $\eta > 0$ such that the (unnormalized) L^∞-norm of the three-dimensional discrepancy function, i.e. the (unnormalized) star discrepancy D_N^*, is bounded from below by $D_N^* \geq c(\log N)^{1+\eta}$, for all $N \in \mathbb{N}$ sufficiently large, where $c > 0$ is some constant independent of N. This paper builds upon their methods to verify that the above result holds with $\eta < 1/(32 + 4\sqrt{41}) \approx 0.017357\ldots$

Keywords: Uniform distribution; Discrepancy; Number theory

1. Introduction and statement of the result

Suppose we are given a set $\mathcal{P}_N$ consisting of N points in the d-dimensional unit cube. We intend to investigate how *well* this set is distributed in $[0, 1)^d$. To this end we introduce the *discrepancy function*

$$D_N(x) := N\lambda_d([0, x)) - \#(\mathcal{P}_N \cap [0, x)), \quad x \in [0, 1)^d,$$

i.e. the difference between the expected and actual number of points of $\mathcal{P}_N$ in $[0, x)$ if we assume uniform distribution. Here, λ_d denotes the d-dimensional Lebesgue measure and we abbreviated $[0, x) = [0, x_1) \times [0, x_2) \times \cdots \times [0, x_d)$ for $x = (x_1, x_2, \ldots, x_d)$. Furthermore, we refer to its L^∞-norm

$$D_N^* := \sup_{x\in[0,1)^d} |D_N(x)|$$

as *star discrepancy*. Notice that, in other literature, this entity often appears as a normalized version, i.e. D_N^*/N.

Over time an extensive theory has evolved around the magnitude of D_N^* in terms of N for arbitrary as well as for specific point sets. See, for instance, the books [7,12,9], just to name a few. Finding the exact order of growth seems to be an intriguingly difficult question and has not yet been solved for dimensions three or higher. In this paper we focus on a lower bound for the star discrepancy of arbitrary sets of N points in the three-dimensional case based on the work of D. Bilyk and M.T. Lacey [4]. More precisely, we show

E-mail address: florian.puchhammer@jku.at.

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Theorem 1. *For all N-point sets in* $[0, 1)^3$*, with N sufficiently large, the star discrepancy satisfies*

$$D_N^* \geq C(\log N)^{1+\eta}, \quad \text{for all } \eta < 1/(32 + 4\sqrt{41}) \approx 0.017357\ldots$$

To the author's best knowledge this is the first quantitative result with respect to η.

It is worth mentioning that the basic inherent ideas reach back to K.F. Roth's seminal paper [15], in which he showed

Theorem 2 (*Roth, 1954*)**.** *We have* $D_N^* \geq \|D_N\|_2 \geq c_d (\log N)^{(d-1)/2}$ *for all* $d \geq 2$.

Although this bound is now known not to be sharp for D_N^* (see Schmidt's theorem below) it was his approach using the system of Haar functions and Haar decompositions which struck a chord at that time and lead to a completely new methodology for proving discrepancy bounds. For a comprehensive survey see [3], for instance.

It took as much as 18 years until a better estimate for D_N^* in the two-dimensional case was discovered by W.M. Schmidt, see [16]:

Theorem 3 (*Schmidt, 1972*)**.** *For* $d = 2$ *we have* $D_N^* \geq C \log N$.

This bound is even known to be sharp. Later, in 1981, G. Halász managed to give a proof of Schmidt's result by refining Roth's approach via introducing special auxiliary functions, namely Riesz products, and using duality, see [8]. Both, Roth's and Halász' proofs are also to be found in [12]. Unfortunately, Halász' methods are not directly applicable to higher dimensions, due to a shortfall of certain orthogonality properties.

This shortfall leads us to yet another main ingredient of the proof of Bilyk and Lacey as well as of this paper. In [2] J. Beck laid the groundwork for combining Halász' approach to graph theory and probability theory in three dimensions. He thereby gave the first improvement to Roth's bound by a double logarithmic factor in this case. In fact, he proved the following theorem.

Theorem 4 (*Beck, 1989*)**.** *For all N-point sets in* $[0, 1)^3$ *and for all* $\varepsilon > 0$ *we have*

$$D_N^* \geq C_\varepsilon \log N \cdot (\log \log N)^{1/8-\varepsilon}.$$

For the sake of completeness one needs to add that an analogue of Theorem 1 for arbitrary dimension $d \geq 4$ was proven by Bilyk and Lacey together with A. Vagharshakyan in [5]. Within their paper they showed that the exponent of the logarithm in Roth's theorem can be increased to $(d-1)/2 + \eta_d$ with an (unspecified) $\eta_d > 0$. Due to the transition to higher dimensions and to simplification reasons several arguments were refined and the overall strategy was slightly changed in comparison to the three-dimensional case. Apart from the increasing combinatorial effort this is one of the main reasons why the same line of reasoning as in the proof of Theorem 1 would not (yet) work in higher dimensions. This might be an interesting subject to be investigated in the future.

The author would also like to mention that a new proof for the lower bound of the star discrepancy of the first N points of a sequence in the unit interval has recently been discovered by G. Larcher, see [10], and has been slightly improved upon in [11], which transfers to two-dimensional point sets by a result from [9].

The second section is dedicated to briefly describe the main ideas of Halász' proof of Theorem 3 as well as to explain why his strategy cannot be directly extended to higher dimensions. This serves as an incentive to present the result of Bilyk and Lacey, i.e. Theorem 1 without the specific bound for η, in Section 3, as they incorporate these ideas and provide the tools to fill the aforementioned gaps. We focus on one of these tools, the so-called *Littlewood–Paley inequalities*, in Section 4 since they play an integral role in our proof. Finally, in Section 5, we carefully estimate the L^1-norm of a certain auxiliary function $\Psi^\neg$ which already appeared in [4]. This, in turn, contributes the crucial bound for η and thus completes the proof of Theorem 1.

2. Halász' proof of Theorem 3

The essential idea behind this proof is to choose an auxiliary function Φ in such a way that it is complicated enough to recapture the overall structure of D_N well, while, on the other hand, it remains relatively easy to handle. More precisely, one constructs Φ such that $\|\Phi\|_1 \leq 2$ and $\langle \Phi, D_N \rangle \geq c \log N$ for some $c > 0$ since then, by duality,

$$2D_N^* = 2\|D_N\|_\infty \geq \langle \Phi, D_N \rangle \geq c \log N.$$