Operations Research Letters 45 (2017) 533–537

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

# Lower semicontinuity of solution mappings for parametric fixed point problems with applications

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ARTICLE INFO

*Article history:*
Received 25 October 2016
Received in revised form 15 August 2017
Accepted 15 August 2017
Available online 22 August 2017

*Keywords:*
Parametric fixed point problem
Solution mapping
Lower semicontinuity
Generalized Stackelberg equilibrium problem
Parametric vector quasiequilibrium problem

ABSTRACT

In this paper, we establish the lower semicontinuity of the solution mapping and the approximate solution mapping for parametric fixed point problems under some suitable conditions. As applications, the lower semicontinuity result applies to the parametric vector quasi-equilibrium problem, and allows to prove the existence of solutions for generalized Stackelberg games.



## 1. Introduction

The semicontinuity of solution mappings of vector equilibrium problems has been investigated by several authors, see [1–4,6,9,12–14,16,17] and the references therein. Recently, in order to show the semicontinuity of the solution mappings for the parametric (vector) quasi-equilibrium problems, all the solution mappings of the parametric fixed point problems are assumed to be lower semicontinuous in the literature [1–3]. We note that in the literature mentioned above, the authors have not given any conditions to guarantee the lower semicontinuity of the solution mappings of the parametric fixed point problems. On the other hand, it is difficult to obtain the explicit solutions for some real problems when the data concerned with the problems are perturbed by noise and so the mathematical models are usually solved by numerical methods for approximating the exact solutions. Therefore, one natural question is: can we provide conditions ensuring the lower semicontinuity of the (approximate) solution mappings?

The main purpose of this paper is to make a new attempt to establish the lower semicontinuity of the solution mapping and the approximate solution mapping for parametric fixed point problems under suitable conditions. As applications, the lower semicontinuity result applies to the parametric vector quasi-equilibrium problem, and allows to prove the existence of solutions for generalized Stackelberg games.

## 2. Preliminaries

Throughout this paper, unless otherwise specified, let $\Lambda$ and $X$ be two normed vector spaces, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}_+^0 = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Let $A$ be a nonempty subset of $X$ and $T : A \times \Lambda \to 2^A$ be a set-valued mapping. For $\lambda \in \Lambda$, we consider the following parametric fixed point problem consisting of finding $x_0 \in A$ such that

(PFPP) $\quad x_0 \in T(x_0, \lambda)$.

For $\lambda \in \Lambda$, let $S(\lambda)$ denote the set of all solutions of (PFPP), i.e.

$$S(\lambda) = \{x \in A : x \in T(x, \lambda)\}.$$

For $(\lambda, \varepsilon) \in \Lambda \times \mathbb{R}_+$, let $E(\lambda, \varepsilon)$ denote the set of all $\varepsilon$-approximate solutions of (PFPP), i.e.

$$E(\lambda, \varepsilon) = \{x \in A : d(x, T(x, \lambda)) \leq \varepsilon\},$$

where $d(x, T(x, \lambda)) = \inf_{y \in T(x,\lambda)} d(x, y)$ and $d(x, y) = \|x - y\|$.

Denote the boundary of $D$ by $\partial D$, the complement of $D$ by $D^c$, the closure of $D$ by cl$D$ and the interior of $D$ by int$D$.

**Definition 2.1** (*[15]*). A nonempty convex subset $D$ of $X$ is said to be rotund if the boundary of $D$ does not contain line segments,

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http://dx.doi.org/10.1016/j.orl.2017.08.011

i.e., for any $x_1, x_2 \in D$ with $x_1 \neq x_2$, $(x_1, x_2) \cap (\partial D)^c \neq \emptyset$, where $(x_1, x_2) = \{\lambda x_1 + (1-\lambda) x_2 : \lambda \in (0,1)\}$.

**Remark 2.1.** Let $D$ be a nonempty convex subset of $X$. Then it is easy to see that $D$ is rotund if and only if for any $x_1, x_2 \in D$ with $x_1 \neq x_2$, there exists $\lambda_0 \in (0,1)$ such that $\lambda_0 x_1 + (1-\lambda_0) x_2 \in \mathrm{int} D$. Let $D = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}$. Then it is clear that $D$ is rotund.

**Definition 2.2.** Let $\Delta$ and $\Delta_1$ be two topological vector spaces. A set-valued mapping $\Phi : \Delta \to 2^{\Delta_1}$ is said to be

(i) upper semicontinuous (u.s.c.) at $u_0 \in \Delta$ if, for any neighborhood $V$ of $\Phi(u_0)$, there exists a neighborhood $U(u_0)$ of $u_0$ such that for every $u \in U(u_0)$, $\Phi(u) \subseteq V$.
(ii) lower semicontinuous (l.s.c.) at $u_0 \in \Delta$ if, for any $x \in \Phi(u_0)$ and any neighborhood $V$ of $x$, there exists a neighborhood $U(u_0)$ of $u_0$ such that for every $u \in U(u_0)$, $\Phi(u) \cap V \neq \emptyset$.
(iii) Hausdorff lower semicontinuous (H-l.s.c.) at $u_0 \in T$ if, for any neighborhood $V$ of $0 \in T_1$, there exists a neighborhood $U(u_0)$ of $u_0$ such that for every $u \in U(u_0)$, $G(u_0) \subseteq G(u) + V$.
(iv) convex if, the graph of $\Phi$, i.e., $Graph(\Phi) := \{(x,y) \in \Delta \times \Delta_1 : y \in \Phi(x)\}$ is a convex set in $\Delta \times \Delta_1$.
(v) rotund if, $Graph(\Phi)$ is convex and for any $(x_1, y_1), (x_2, y_2) \in Graph(\Phi)$ with $x_1 \neq x_2, y_1 \neq y_2$, we have

$$((x_1, y_1), (x_2, y_2)) \cap (\partial Graph(\Phi))^c \neq \emptyset,$$

where $((x_1, y_1), (x_2, y_2)) = \{\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) : \lambda \in (0,1)\}$.

We say that $\Phi$ is u.s.c. and l.s.c. on $\Delta$ if it is u.s.c. and l.s.c. at each point $u \in \Delta$, respectively. $\Phi$ is continuous on $\Delta$ if it is both u.s.c. and l.s.c. on $\Delta$.

**Remark 2.2.** Obviously, if $\Phi : \Delta \to 2^{\Delta_1}$ is convex, then $\Phi(x)$ is a convex set for any $x \in \Delta$.

**Lemma 2.1** ([5]). *A set-valued mapping $\Phi : \Delta \to 2^{\Delta_1}$ is l.s.c. at $u_0 \in \Delta$ if and only if for any sequence $\{u_n\} \subseteq \Delta$ with $u_n \to u_0$ and for any $x_0 \in \Phi(u_0)$, there exists $x_n \in \Phi(u_n)$ such that $x_n \to x_0$.*

**Lemma 2.2** ([10]). *Let $\Phi : \Delta \to 2^{\Delta_1}$ be a set-valued mapping. For any given $u_0 \in \Delta$, if $\Phi(u_0)$ is compact, then $\Phi$ is u.s.c. at $u_0 \in \Delta$ if and only if for any sequence $\{u_n\} \subseteq \Delta$ with $u_n \to u_0$ and for any $x_n \in \Phi(u_n)$, there exist $x_0 \in \Phi(u_0)$ and a subsequence $\left\{x_{n_k}\right\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$.*

**Lemma 2.3** (*Kakutani–Fan–Glicksberg Fixed Point Theorem [7,8]*). *Let $K$ be a nonempty compact convex subset of a locally convex Hausdorff topological vector space $X$ and let $F : K \to 2^K$ be a u.s.c. set-valued mapping with nonempty compact convex values. Then there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.*

## 3. The main results

**Lemma 3.1.** *Let $X$ be a reflexive Banach space, $B$ be the closed unit ball of $X$ and $A$ be a nonempty closed convex subset of $X$. For given $\delta > 0$, if $a + \delta B \subseteq A + \delta B$, then $a \in A$.*

**Proof.** Suppose on the contrary that $a \notin A$. Since $A$ is closed, one has

$$d(a, A) = \inf_{y \in A} \|a - y\| > 0.$$

Noting that $X$ is a reflexive Banach space and $A$ is a nonempty closed convex subset of $X$, there exists $\beta \in A$ such that

$$d(a, A) = \inf_{y \in A} \|a - y\| = \|a - \beta\| > 0. \tag{1}$$

Let $\lambda = \frac{\|a-\beta\|}{\delta + \|a-\beta\|}$. We choose $h \in X$ such that $a = \lambda h + (1-\lambda)\beta$. Then $h = \frac{a}{\lambda} + \beta - \frac{\beta}{\lambda}$.

We claim that $\|h - y\| \geq \|h - \beta\|$ for any $y \in A$. It follows from (1) that

$$\|a - y\| \geq \|a - \beta\|, \quad \forall y \in A. \tag{2}$$

For any $y \in A$, since $y, \beta \in A$ and $A$ is convex, we have $\lambda y + (1-\lambda)\beta \in A$. By (2), we know that $\|a - (\lambda y + (1-\lambda)\beta)\| \geq \|a - \beta\|$ and so

$$\|h - y\| \geq \|h - \beta\|, \quad \forall y \in A. \tag{3}$$

On the other hand,

$$\|h - a\| = \left\| \frac{a}{\lambda} + \beta - \frac{\beta}{\lambda} - a \right\| = \left(\frac{1}{\lambda} - 1\right) \|a - \beta\| = \delta$$

and so $h \in a + \delta B$. Noting that (3) and

$$\|h - \beta\| = \left\| \frac{a}{\lambda} + \beta - \frac{\beta}{\lambda} - \beta \right\| = \frac{1}{\lambda} \|a - \beta\| = \delta + \|a - \beta\| > \delta,$$

we have $h \notin A + \delta B$ and so $a + \delta B \not\subseteq A + \delta B$, which contradicts the assumption that $a + \delta B \subseteq A + \delta B$. This completes the proof. □

**Theorem 3.1.** *Let $\lambda_0 \in \Lambda$ and $A$ be a nonempty compact convex subset of a reflexive Banach space $X$. Assume that $T(\cdot, \lambda_0)$ is rotund and $T(\cdot,\cdot)$ is continuous on $A \times \{\lambda_0\}$ with nonempty closed convex values. Then $S(\cdot)$ is l.s.c. at $\lambda_0$.*

**Proof.** Suppose on the contrary that $S(\cdot)$ is not l.s.c. at $\lambda_0$. Then there exist a point $x_0 \in S(\lambda_0)$, a neighborhood $W_0$ of $0 \in X$ and a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that

$$(x_0 + W_0) \cap S(\lambda_n) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{4}$$

There are two cases to be considered.

Case 1. $S(\lambda_0)$ is a singleton. For $x_n \in S(\lambda_n)$, one has

$$x_n \in T(x_n, \lambda_n), \quad \forall n \in \mathbb{N}. \tag{5}$$

Since $x_n \in A$ and $A$ is compact, without loss of generality, we can assume that $x_n \to \bar{x} \in A$. Noting that $T(\cdot,\cdot)$ is u.s.c. at $(\bar{x}, \lambda_0)$, it follows from Lemma 2.2 and (5) that there exist a point $x' \in T(\bar{x}, \lambda_0)$ and a subsequence $\left\{x_{n_k}\right\}$ of $\{x_n\}$ such that $x_{n_k} \to x'$. By $x_n \to \bar{x}$, we know that $\bar{x} = x'$ and so $\bar{x} = x' \in T(\bar{x}, \lambda_0)$. This means that $\bar{x} \in S(\lambda_0)$. Noting that $S(\lambda_0)$ is a singleton, we have $\bar{x} = x_0$ and so $x_n \to \bar{x} = x_0$. Thus, $x_n \in x_0 + W_0$ for $n$ large enough. This together with $x_n \in S(\lambda_n)$ implies that $(x_0 + W_0) \cap S(\lambda_n) \neq \emptyset$ for $n$ large enough, which contradicts (4).

Case 2. $S(\lambda_0)$ is not a singleton. Then there exists $x^* \in S(\lambda_0)$ such that $x^* \neq x_0$. Since $x^*, x_0 \in S(\lambda_0)$, we know that $x^* \in T(x^*, \lambda_0)$ and $x_0 \in T(x_0, \lambda_0)$. Thus, $(x^*, x^*), (x_0, x_0) \in Graph(T(\cdot, \lambda_0))$. Let

$$x(t) = tx^* + (1-t)x_0, \quad \forall t \in [0,1].$$

Then it is clear that $x(t) \in A$. Since $Graph(T(\cdot, \lambda_0))$ is rotund, we can find $t_0 \in (0,1)$ such that

$$x(t_0) \in x_0 + W_0 \tag{6}$$

and

$$(x(t_0), x(t_0)) \in \mathrm{int}(Graph(T(\cdot, \lambda_0))). \tag{7}$$

It follows from (7) that there exists a constant $\delta > 0$ such that

$$(x(t_0), x(t_0)) + \delta B \times \delta B \in Graph(T(\cdot, \lambda_0)),$$

where $B$ is the closed unit ball in $X$. This shows that

$$x(t_0) + \delta B \subseteq T(x(t_0), \lambda_0). \tag{8}$$