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Sensitivity analysis of the value function for nonsmooth optimization problems

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ABSTRACT

In this paper we present sensitivity analysis for a nonsmooth optimization problem with equality and inequality constraints. A necessary optimality condition, based on the convexificators, under the local error bound constraint qualification is derived. Then, we employ them to establish upper estimates for Fréchet and limiting subdifferentials of the value function. Furthermore, we present sufficient conditions for Lipschitzness of the value function at the point of interest. Also, some examples are provided to clarify our results.

1. Introduction

In this paper, we consider the following nonlinear optimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_j(x) \leqslant 0,\ j \in J := \{1, \dots, p\}, \\ & h_i(x) = 0,\ i \in I := \{1, \dots, q\}, \\ & x \in X, \end{array} \tag{NLP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, $g_j, h_i : \mathbb{R}^n \to \mathbb{R}$ for all $i \in I, j \in J$ are continuous functions and $X \subset \mathbb{R}^n$ is a closed and convex set.

Recently, the idea of convexificators has been used to extend, unify and sharpen the results in various aspects of optimization. The convexificators serve a useful tool for treating problems with continuous functions. A whole machinery of calculus is devoted for convexificators and mostly gives sharp results for locally Lipschitz functions as the Clarke subdifferential may contain the closed convex hull of a convexificator; see, e.g., [5–7,12]. In the context of optimality conditions and constraint qualifications (CQs), various results that use convexificators have been developed; see [7,11] and references therein.

In this work, we apply the concept of convexificators to pursue a bipartite goal. Our first aim is to investigate the necessary optimality conditions for NLP under the well-known local error bound constraint qualification. In the second one, these optimality conditions are applied to derive new results on characteristics of some subdifferentials of the value function regarding the following parametric optimization problem:

$$\begin{array}{ll} \min & f(x, y) \\ \text{s.t.} & g_j(x, y) \leqslant 0,\ j \in J, \\ & h_i(x, y) = 0,\ i \in I, \\ & x \in X, \end{array} \tag{1}$$

where $f, g_j, h_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, for all $i \in I, j \in J$. Value functions have an effective role in variational analysis, optimization, control theory, and various applications [16]. It is well known that value functions are intrinsically nonsmooth even for smooth f, g and h. Thus, one needs to know about the subdifferential of the value function in order to obtain information on sensitivity and stability of optimization and control problems with respect to parameter perturbations. The suitable subdifferentials constructions of f, g and h give information including evaluating subdifferentials of the value function at the desired point. There is a large amount of literature for value functions; see, e.g. [1,3,4,9,10,14,18,20]. The authors in [18] derived some results for computing and estimating the Fréchet and limiting subdifferentials of value functions of a class of general optimization problems with abstract set-valued mapping constraint with smooth and nonsmooth data. The sensitivity of the two-level value function of an optimistic bilevel program with continuously differentiable data was investigated in [4]. To the best of our knowledge, the key requirements for subdifferential estimates and sensitivity analysis of value functions are the classical Mangasarian–Fromovitz CQ and its nonsmooth extensions.

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To our knowledge for the first time, using the notion of convexificators, we present upper estimates for Fréchet and limiting subdifferentials of the NLP value function. We also give verifiable conditions for its local Lipschitz continuity. Notice that the main tool used to achieve these aims is the existence of a local error bound.

It is worth mentioning that the results in this study are established regardless of which convexificator is used and thus can be applied to a large class of subdifferentials; see Remark 2.

The rest of the paper is organized as follows. Section 2 presents basic definitions and preliminaries from generalized differentiation widely used in the sequel. In Section 3, by using the idea of convexificators, the Karush–Kuhn–Tucker (KKT) necessary optimality conditions for NLP are derived. Then upper estimates for Fréchet and limiting subdifferentials of the NLP value function are established as well as the sufficient conditions for its local Lipschitz continuity. Some examples are provided to illustrate the results.

2. Preliminaries

In this section, we recall some classical notions and basic results from nonsmooth analysis needed in what follows. Our notation is basically standard. We denote by $\|\cdot\|$ the Euclidean norm in n-dimensional space $\mathbb{R}^n$ and also

$$\mathbb{R}^n_+ := \{(x_1,\dots,x_n) : x_i \geqslant 0 \text{ for all } i = 1,\dots,n\}.$$

The inner product between two vectors $x, y \in \mathbb{R}^n$ is defined by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. The closed unit ball of $\mathbb{R}^n$, denoted by $\mathbb{B}_n$, is defined by $\mathbb{B}_n := \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}$. For a given set $S \subseteq \mathbb{R}^n$, the distance function $d_S : \mathbb{R}^n \to \mathbb{R}_+$ is defined by $d_S(x) := \inf_{y\in S}\|y - x\|$, and also cl$S$ and co$S$ stand for the closure and convex hull of S, respectively. Now, we state the following lemma which is useful in the sequel.

Lemma 1. *Let S and T be two nonempty subsets of $\mathbb{R}^n$. Then the following hold:*

(i) $\mathrm{cocl}S \subset \mathrm{clco}S$.
(ii) $\mathrm{co}(S \cup T) = \bigcup_{0\leqslant t\leqslant 1} t\,\mathrm{co}\,S + (1-t)\mathrm{co}\,T$.

Proof. The proof of part (i) is trivial and part (ii) follows immediately from [19, Theorem 3.3]. □

Considering the set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we say that M is upper semicontinuous (u.s.c.) at x if for each $\varepsilon > 0$, there exists a positive scalar δ such that

$$M(x + \delta\mathbb{B}_n) \subseteq M(x) + \varepsilon\mathbb{B}_m.$$

Next let us present some of the basic concepts of generalized differentiation from [3,16]. We start with the tangent and normal cones to convex sets. Let S is a convex set and $x \in \mathrm{cl}S$, then the tangent and normal cones to S at x are defined, respectively, by

$$T(x; S) := \mathrm{cl}\{\lambda(s - x) : \lambda \geqslant 0, s \in S\},$$
$$N(x; S) := \{\xi : \langle \xi, s - x\rangle \leqslant 0, \forall s \in S\}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function at x. The Fréchet (regular) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\hat{\partial} f(x) := \left\{\xi \in \mathbb{R}^n \ : \ \liminf_{x'\to x} \frac{f(x') - f(x) - \langle \xi, x' - x\rangle}{\|x' - x\|} \geqslant 0\right\}.$$

The limiting (Mordukhovich) subdifferential of f at x is given by

$$\partial_L f(x) := \{\xi \in \mathbb{R}^n \ : \ \exists\, x_k \xrightarrow{f} x, \text{ and } \xi_k \in \hat{\partial} f(x_k) \text{ with } \xi_k \to \xi\},$$

and the singular limiting subdifferential of f at x is defined by

$$\partial^\infty f(x)$$
$$:= \{\xi \in \mathbb{R}^n \ : \ \exists\, x_k \xrightarrow{f} x,\ t_k \downarrow 0, \text{ and } \xi_k \in \hat{\partial} f(x_k) \text{ with } t_k\xi_k \to \xi\}.$$

Remark 1. Let us emphasize that f is Lipschitz around x if and only if $\partial^\infty f(x) = \{0\}$.

If f is assumed to be locally Lipschitz near x, then for each $u \in \mathbb{R}^n$,

$$f^\circ(x; u) := \limsup_{\substack{x'\to x \\ t\downarrow 0}} \frac{f(x' + tu) - f(x')}{t},$$

is known as the Clarke generalized derivative of f at x with respect to u. The Clarke subdifferential of f at x is defined by

$$\partial_C f(x) := \{\xi \in \mathbb{R}^n : \langle \xi, u\rangle \leqslant f^\circ(x; u),\ \forall u \in \mathbb{R}^n\}.$$

We continue by recalling the notion of the convexificator and some of its important properties from [12]. The lower and upper Dini derivatives of a given function $f : \mathbb{R}^n \to \mathbb{R}$ at x in the direction $u \in \mathbb{R}^n$ are given, respectively, as

$$f^-(x; u) := \liminf_{t\downarrow 0} \frac{f(x + tu) - f(x)}{t},$$

and

$$f^+(x; u) := \limsup_{t\downarrow 0} \frac{f(x + tu) - f(x)}{t}.$$

Definition 1.

- The function f is said to have an upper convexificator at $x \in \mathbb{R}^n$ if there is a closed set $\partial f(x) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,
$$f^-(x; u) \leqslant \sup_{\xi\in\partial f(x)} \langle \xi, u\rangle .$$
- The function f is said to have a lower convexificator at $x \in \mathbb{R}^n$ if there is a closed set $\partial f(x) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,
$$f^+(x; u) \geqslant \inf_{\xi\in\partial f(x)} \langle \xi, u\rangle .$$
- A closed set $\partial f(x) \subset \mathbb{R}^n$ is said to be a convexificator of function f at x if it is both upper and lower convexificators for f at x.
- The function f is said to have an upper regular convexificator at $x \in \mathbb{R}^n$ if there is a closed set $\partial f(x) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,
$$f^+(x; u) = \sup_{\xi\in\partial f(x)} \langle \xi, u\rangle .$$

Remark 2. It is worth to mention that if f is continuous and Gâteaux differentiable at x, then $\{\nabla f(x)\}$ is a convexificator of f at x and if f is a locally Lipschitz function, then the Clarke $\partial_C f$ [3], Michel–Penot $\partial_{MP} f$ [15], limiting $\partial_L f$ [16] and Treiman $\partial_l f$ [21] subdifferentials are examples of convexificators for f.

We proceed this section by giving the following calculus rules for convexificators, that are useful in the sequel. Letting $A \subset \{F : \mathbb{R}^n \to \mathbb{R}^m\}$ and $B \subset \{G : \mathbb{R}^l \to \mathbb{R}^n\}$, we denote by $A \circ B$ the set of all composition functions $F \circ G$ with $F \in A$ and $G \in B$.

Proposition 1.

(i) *Let $\partial f(x)$ be a convexificator of f at x, then $\alpha\partial f(x)$ is a convexificator of αf at x for every $\alpha \in \mathbb{R}$.*
(ii) *Assume that $\partial f(x)$ is an upper convexificator of f and g admits upper regular convexificator $\partial g(x)$ at x, then $\partial f(x) + \partial g(x)$ is an upper convexificator of $f + g$ at x.*
(iii) *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. If ∂f and ∂g are bounded u.s.c. convexificators of f and g at x and $f(x)$, respectively. Then the set $\partial g(f(x))\circ\partial f(x)$ is a convexificator of the composite function $g \circ f$ at x.*