Operations Research Letters 45 (2017) 362–365

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

On the existence of ideal Nash equilibria in discontinuous games with infinite criteria

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ARTICLE INFO

Article history:
Received 3 January 2017
Received in revised form 15 May 2017
Accepted 15 May 2017
Available online 20 May 2017

Keywords:
Infinite-criteria games
Ideal Nash equilibria
Discontinuous games
Pseudocontinuity
Existence

ABSTRACT

In this paper, we introduce the notion of ideal Nash equilibria in infinite-criteria games. Applying the maximal element theorem, we provide an existence theorem of ideal Nash equilibria in infinite-criteria games with discontinuous payoff functions. We further give a necessary and sufficient condition for the existence of ideal Nash equilibria.

1. Introduction

In noncooperative games, the payoff of a player reflecting the desirability of an outcome to him is usually unidimensional. In many practical problems, however, the players' decisions are often guided by multiple goals/criteria. Multicriteria game, in which a player's objective is represented by partial-ordered vectors, was first introduced by Blackwell [4]. Some subsequent studies concern the solution concepts of multicriteria games (cf. [4,7,13,23]). The notion of Pareto-Nash equilibria proposed by Shapley and Rigby [18] is the fundamental equilibrium concept in multicriteria games as a generalization of Nash equilibrium in single-criterion games [11]. Voorneveld et al. [22] introduced a new concept of ideal Nash equilibria for finite-criteria games, which enables a player to maximize all his criteria when the other players choose their ideal Nash equilibrium strategies. Radjef and Fahem [14] provided an existence theorem of ideal Nash equilibria with the aid of a maximal element theorem due to [6].

The existing literature on this issue concerns mainly about games with finite criteria and continuous payoffs. Games in many important economic models, such as those in Bertrand [3], Hotelling [8], Dasgupta and Maskin [5], and Jackson [9], have discontinuous payoffs. Studies on the theory of equilibrium existence in single-criterion or finite-criteria games with discontinuous payoffs are abundant (cf. Dasgupta and Maskin [5]; Simon [19]; Simon and Zame [20]; Reny [15,16]; Tian [21]; Nessah and Tian [12]). To our knowledge, discontinuous games with infinitely many criteria have received relatively little attention in game-theoretic literature. This kind of game is, however, important and can be applied to a variety of significant practical problems. A representative agent in a multi-criteria game is usually viewed as a multi-member organization, in which each criterion corresponds to the goal of a member. In various environments, there are infinitely many members in organizations, so the number of criteria may be infinite.

Next, we illustrate the motivation of this paper with an electoral game. Two parties ($i = 1, 2$) compete for a set of voters $[0, 1]$ by choosing simultaneously their policy platforms P_i within the interval $[0, 1]$. Each voter has a certain "bliss point" $x \in [0, 1]$. This could be regarded as his ideological position along a Left–Right political system: anyone could be an ultra-liberal and be on the far left of the spectrum or be very conservative and be on the right. Voters are assumed to have single-peaked preferences. If party i is elected and policy P_i is implemented, then voter x obtains a payoff $u(x, P_i) = A - (x - P_i)^2$, where A is a sufficiently large number guaranteeing a positive payoff. Every voter casts his ballot for the party with a closer platform. The voter who feels indifferent between two parties will abstain. We denote by $S_i(P_1, P_2) \equiv \{x \in [0, 1] : |x - P_i| < |x - P_{-i}|\}$ the set of strict supporters of party i. Only one party will eventually take the office. By majority rule, party i will win the election if $m(S_i(P_1, P_2)) > m(S_{-i}(P_1, P_2))$ ($m(\cdot)$ is the measure of a set); two parties win with even chances if $m(S_1(P_1, P_2)) = m(S_2(P_1, P_2))$. Once a party wins the election, he represents and aims at maximizing the interest of every voter (not

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http://dx.doi.org/10.1016/j.orl.2017.05.004
0167-6377/

merely his supporters), while the losing party does not care about interest of any voter (not even his supporters). Therefore, party i has the following payoff:

$$V_i(P_1,P_2) = \begin{cases} (u(x,P_i))_{x\in[0,1]} & \text{if} \quad m(S_i(P_1,P_2)) > m(S_{-i}(P_1,P_2)) \\ 1/2\times(u(x,P_i))_{x\in[0,1]} & \text{if} \quad m(S_i(P_1,P_2)) = m(S_{-i}(P_1,P_2)) \\ (0)_{x\in[0,1]} & \text{if} \quad m(S_i(P_1,P_2)) < m(S_{-i}(P_1,P_2)). \end{cases}$$

It is obvious that both parties have infinitely many criteria, and the payoff functions V_1, V_2 are discontinuous on $[0,1]\times[0,1]$. To see this, note that $V_1(P,P) = V_2(P,P) = 1/2\times(u(x,P))_{x\in[0,1]}$ if $P < 1/2$; $V_1(P,P+\epsilon) = (0)_{x\in[0,1]}$, $V_2(P,P+\epsilon) = (u(x,P+\epsilon))_{x\in[0,1]}$ if $\epsilon \in (0,1-2P)$. So V_1 and V_2 are both discontinuous at (P,P). We define a partially ordered preference $\succcurlyeq_i$ of a party i as follows: $V\succcurlyeq_i V'$ if $(V)_k \geq (V')_k, \forall k\in[0,1]$; $V\succ_i V'$ if $(V)_k \geq (V')_k, \forall k \in [0,1]$ and strict inequality holds for at least one $k\in[0,1]$, where $(V)_k$ denotes the kth entry of the infinite-dimensional payoff vector V. Under $\succcurlyeq_i$, $i = 1,2$, we find that the unique Nash equilibrium is $P_1 = P_2 = 1/2$. Suppose it is not the case. We assume, without loss of generality, that $P_1 < P_2$. If $P_1+P_2 = 1$, then the election is tied, both parties will deviate and choose a policy a bit closer to 1/2. To see this, note that $V_1(P_1+\epsilon, P_2) = (u(x,P_1+\epsilon))_{x\in[0,1]}\succ_1 V_1(P_1,P_2) = 1/2\times(u(x,P_1))_{x\in[0,1]}$ and $V_2(P_1,P_2-\epsilon)\succ_2 V_2(P_1,P_2)$ for some small $\epsilon > 0$. If $P_1+P_2 \neq 1$, we have a non-tied outcome, then the loser will obviously deviate. Therefore, we must have $P_1 = P_2$ in the equilibrium. Next, we proceed to show that $P_1 = P_2 = 1/2$. Suppose that $P_1 = P_2 \neq 1/2$, then any party has incentive to move slightly toward 1/2. He will defeat his rival and will improve the interests of all voters by doing so. Summarizing, we have the unique Nash equilibrium $P_1 = P_2 = 1/2$. Note that in our model, a party aims at maximizing the infinite-dimensional payoff vector of all voters rather than cares only about winning the election. But the Median Voter Theorem (MVT) still applies. However, it is worth noting that the applicability of MVT in this setup crucially depends on the utility function adopted. If we use other form, say $u(x,P_i) = 1/|x-P_i|$, then any $(P_1,P_2)\in\{(x,y)\in[0,1]^2 | x+y = 1\}$ forms a Nash equilibrium, since for every party, any unilateral deviation will not be a Pareto improvement for all voters. However, the existing game-theoretic literature still lacks a general methodology for studying the existence of equilibria in this kind of games. In this paper, we fill this gap by providing the existence theorem of ideal Nash equilibria in noncooperative games with pseudocontinuous payoff functions and infinite criteria.

The remainder of the paper is organized as follows. In Section 2, we give some preliminary definitions; Section 3 gives the main results.

2. Preliminaries and definitions

A game with infinite criteria is a list $\tau = \langle I, X_i, Y, f_i, G_i\rangle$, where $I = \{1,\ldots,n\}$ is the set of players; X_i is the set of actions for player i, $X = \prod_{i\in I} X_i$, $X_{-i} = \prod_{j\neq i} X_j$, Y is the set of criteria; $G_i : X \rightrightarrows Y$ is the feasible-criterion mapping of player i; for each $y\in Y$, $f_i(y,\cdot) : X\to\mathbb{R}$ is the payoff function of player i with respect to the criterion y.

Definition 2.1. A strategy profile $x^*\in X$ is

- a weakly efficient Nash equilibrium of τ if for each $i\in I$ and each $z_i\in X_i$, there exists $y\in G_i(x^*)$ such that

  $f_i(y,z_i,x^*_{-i}) \leq f_i(y,x^*)$;

- an efficient Nash equilibrium of τ if for each $i\in I$ and each $z_i\in X_i$, there exists $y\in G_i(x^*)$ such that

  $f_i(y,z_i,x^*_{-i}) < f_i(y,x^*)$;

- an ideal Nash equilibrium of τ if for each $i\in I$ and each $z_i\in X_i$,

  $f_i(y,z_i,x^*_{-i}) \leq f_i(y,x^*), \quad \forall y\in G_i(x^*)$.

Remark 2.1. (1) If $G_i(x) = \{1,\ldots,r(i)\}$ for all $x\in X$ and for each $i\in I$, our ideal Nash equilibria, efficient Nash equilibria and weakly efficient Nash equilibria coincide with the concepts of Definitions 3 and 4 in [14]. (2) If Y is a singleton set, then an ideal Nash equilibrium is a classical Nash equilibrium, while an efficient Nash equilibrium is a strict Nash equilibrium.

We next recall a weaker concept of continuity, introduced in [10].

Definition 2.2. Let X be a topological space. A function $f : X\to\mathbb{R}$ is

(i) upper pseudocontinuous at $z_0\in X$ if for all $z\in X$ such that $f(z_0) < f(z)$, we have $\limsup_{y\to z_0} f(y) < f(z)$;

(ii) upper pseudocontinuous on X if it is upper pseudocontinuous at all $z\in X$;

(iii) lower pseudocontinuous at $z_0\in X$ if $-f$ is upper pseudocontinuous at $z_0\in X$;

(iv) lower pseudocontinuous on X if it is lower pseudocontinuous at all $z\in X$;

(v) pseudocontinuous on X if it is upper and lower pseudocontinuous on X.

Remark 2.2. The concept of upper (resp. lower) pseudocontinuity is strictly weaker than upper (resp. lower) semicontinuity. The converse is false. See the following example.

Example 2.1. Let $f : [0,2]\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & x\in[0,1] \\ x+1 & x\in(1,2]. \end{cases}$$

It is easy to verify that f is not upper semicontinuous at $x_0 = 1$, but upper pseudocontinuous. Moreover, $f(2) = \max_{x\in[0,2]} f(x)$.

Remark 2.3. Every upper pseudocontinuous function guarantees the existence of maximum points on a compact set. Appendix illustrates this result.

Lemma 2.1 ([17]). *Let X be a Hausdorff topological space. A real-valued function $f : X\to\mathbb{R}$ is pseudocontinuous on X if and only if when $f(x) < f(z)$, there exist an open neighborhood N_x of x and an open neighborhood N_z of z such that $f(x') < f(z')$ for all $x'\in N_x$ and for all $z'\in N_z$.*

We next give an extension of P-quasi-concave-like functions in [2].

Definition 2.3. Let X, Y be topological vector spaces, D a nonempty convex subset of X and $G : X\rightrightarrows Y$ be a set-valued mapping. A function $f : Y\times X\to\mathbb{R}$ is quasi-concave-like with respect to the mapping G if for each $x_1,x_2\in D$, $x\in X$ and $t\in[0,1]$, we have $f(y,tx_1+(1-t)x_2) \geq f(y,x_1)$, $\forall y\in G(x)$ or $f(y,tx_1+(1-t)x_2) \geq f(y,x_2)$, $\forall y\in G(x)$.

Remark 2.4. If $G(x) = \{1,\ldots,k\}$ for all $x\in X$, the quasi-concave-like function with respect to G is equivalent to the $\mathbb{R}^k_+$-quasi-concave-like function.

We recall the maximal element theorem due to [6].

Theorem 2.1 ([6]). *Let I be any index set. Suppose that the following conditions are satisfied: for each $i\in I$,*