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# The economic order decision with continuous dynamic pricing and batch supply

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ABSTRACT

In an infinite horizon inventory and sales model, we show that the seller's unique strategy exhibits increasing prices under general conditions on the revenue function. An increasing discount rate leads to an increase of the time interval between order times, but an increase in batch size has an ambiguous effect.



## 1. Introduction

A seller offers a single product to customers over infinite time, and these customers buy the product according to a deterministic demand function. The seller uses dynamic pricing to maximize his time-discounted profits. The seller is a retailer who can only acquire the good in large bulks, at a fixed cost for every new batch. Our model considers inventory holding costs implicitly by maximizing time-discounted profits, which captures the opportunity cost of the capital when accounting for the inventory holding cost. A typical example is an airline company which plans flight schedules ahead over a certain stretch of time and typically offers tickets at changing, usually increasing, prices.

For overviews of the literature on dynamic pricing in the presence of inventory considerations, see [2,9], and [7]. With only few exceptions, mentioned below, this literature maximizes average profits over a fixed time horizon and assumes that any quantity can be ordered.

A strategy of the seller consists of a specification of order times and sales prices at each point in time. Under concavity of the revenue function, the seller's optimal strategy turns out to be stationary and unique. It is characterized by the time that elapses between any two order moments and an optimal path of increasing prices or, equivalently, increasing prices.

In a comparative statics analysis, we show that the time between two order moments increases with batch cost, a result that confirms intuition. In the case of linear demand, we show that the time between two order moments also increases with the discount rate, but it is not clear whether this is the case for more general demand functions. As to the effect of batch size on the time between order moments, intuition suggests a positive effect. However, even for the case with linear demand functions, we show that it can be both positive and negative.

The works that are closest to our approach are on the one hand [18] and on the other hand [8] and [12]. The first paper considers the maximization of average profits over a fixed time horizon rather than discounted profits over an infinite time horizon. Another important difference with [18] is that we assume the good is ordered in batches of a fixed size. In that respect our approach is similar to [8] and [12], among the few papers in the literature that also consider this case. Contrary to our model of continuous price setting, these two papers consider the case where prices are set at the beginning of each period. Other related works, following the seminal contribution [14], are [17] and [10]. Extensions to multiple interacting players are considered in [15] and [16]. This literature also maximizes discounted profits using dynamic pricing, but deals with a fixed planning horizon and firms which operate under a convex increasing production cost function and choosing a production rate.

Further papers in the literature also assume deterministic demand functions, but consider discrete time models. A notable example is [13] which considers the case with multiple items and demand functions exhibiting seasonality. In [4], the effects of costly

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price adjustment are analyzed, allowing for different costs for price increases versus price decreases.

Still other papers have studied models where demand is stochastic, see [5,6,11], and [3].

In Section 2, we introduce the model. Section 3 provides the optimal sales strategy. Section 4.1 provides the general sensitivity analysis with respect to the parameters of the model, and Section 4.2 analyzes the case of linear demand. Section 5 concludes.

## 2. The model

A manufacturer delivers a non-perishable good in batches of size $S > 0$ for a price $K > 0$. The price $K$ is the total cost for the seller. The seller's inventory level can never become negative, that is, backlogging is not allowed. He can choose when to order new stock and how much he is willing to sell from the stock at every moment in time. Newly ordered stock is delivered instantly. Time is continuous and the time horizon is infinite.

Revenue streams and costs are discounted at a rate of $r > 0$. We assume that $r$ is also equal to his opportunity cost of capital and therefore include inventory holding costs related to the opportunity cost of capital invested in inventories. Other holding costs like the costs of decay and costs related to space and handling of the product should be incorporated explicitly when deemed important. This is a possible extension of the current model.

The non-negative quantity $q(t)$ is the amount of the good the seller decides to supply at time $t$. The resulting function $q$ is assumed to belong to $Q$, the set functions that are piecewise continuous on any finite interval of $[0, \infty)$ and do not have removable discontinuities.

The instantaneous revenue for selling a quantity is given by the continuous function $R$ on $[0, \infty)$. We assume that $R$ is positive on an interval $(0, A)$, is zero on $\{0\} \cup [A, \infty)$, is twice continuously differentiable on $[0, A)$, has a unique maximum at $q^{\mathrm{m}}$ such that $0 < q^{\mathrm{m}} < A$, and is strictly concave on $[0, A]$. Notice that the twice differentiability of $R$ at 0 implies that $R'(0)$ is finite.

Let $X(t)$ be the inventory level of the seller at time $t \geqslant 0$, and let $T_0, T_1, \ldots$ with $T_0 = 0$ be the order moments. Between order moments, stock decreases with rate $q(t)$, and at each order moment it increases with $S$. A *strategy* is a tuple $\sigma = (q, T_1, \ldots)$ such that $q \in Q, T_1, \ldots \in \mathbb{R}$ with $0 < T_1 < \ldots$, and such that $X(t) \geqslant 0$ for all $t \geq 0$. By $\mathcal{S}$, we denote the set of all strategies. The seller therefore faces the following optimal control problem.

$$\max_{(q,T_1,\ldots)\in\mathcal{S}} \sum_{i=0}^{\infty} \left( \int_{T_i}^{T_{i+1}} e^{-rt} R(q(t))dt - e^{-rT_i} K \right)$$
$$\text{subject to} \tag{1}$$
$$X(0) = S,\ \dot{X}(t) = -q(t);\ \text{ for all } i \geqslant 1, X(T_i) = \lim_{t\uparrow T_i} X(t) + S.$$

Observe that $X(t)$ has discontinuities at the points $T_1, T_2$, etc. At these points, $\dot{X}(t)$ is interpreted as the right derivative.

We assume that the seller can make a positive profit on each batch, that is, $K$ is smaller than the maximum discounted revenue that the seller can receive for selling a single batch of size $S$.

## 3. The optimal order path

We start with some useful observations. First, in an optimal strategy the seller will never run out of stock, so $X(t) > 0$ for all $t \geqslant 0$, since otherwise he could simply shift part of his strategy to the moment where he first ran out of stock, and increase his profits due to discounting, contradicting optimality. Second, in an optimal strategy the seller will never order before he runs out of stock, since in such a case he could increase profits by postponing reordering until he runs out of stock, thereby decreasing costs, again due to discounting. Third, in an optimal strategy we have $q(t) \leqslant q^{\mathrm{m}}$ for all $t \geqslant 0$, since otherwise decreasing the offered quantity to $q^{\mathrm{m}}$ both increases instantaneous profit and decreases the cost of ordering new stock due to discounting – since sales speed is reduced, reordering is postponed.

We summarize these observations in the following lemma.

**Lemma 3.1.** *Let $\sigma = (q, T_1, \ldots) \in \mathcal{S}$ be an optimal strategy for problem (1). Then, for all $t \geqslant 0$ and $i \geqslant 1$, we have $X(t) > 0$, $\lim_{t\uparrow T_i} X(t) = 0$, and $q(t) \leqslant q^{\mathrm{m}}$.*

As a step towards solving (1), we first determine the optimal strategy for selling a batch $S$ in a fixed time interval $[0, T]$. As in Lemma 3.1, it is not hard to see that we may assume $T \geqslant T^{\mathrm{m}} = S/q^{\mathrm{m}}$.

Let $Q^T$ be the set of non-negative piecewise continuous functions without removable discontinuities with domain $[0, T]$. The optimal control problem to solve for the case without reordering is

$$\max_{q\in Q^T} \int_0^T e^{-rt} R(q(t))dt$$
$$\text{subject to} \tag{2}$$
$$X(0) = S,\ X(T) = 0,\ \dot{X}(t) = -q(t) \text{ for all } t \in [0, T].$$

Note that we can set $X(T) = 0$ since we are considering the case without reordering.

Problem (2) can be handled by Pontryagin's maximum principle. See [1] for the (simple) derivation of the following proposition.

**Proposition 3.2.** (a) *Problem (2) has a solution. If $q^* \in Q^T$ is such a solution, then $q^*$ is continuous on $[0, T]$ and there is $c^* \geqslant 0$ and $t^* \in [T^{\mathrm{m}}, T]$ such that*

$$R'(q^*(t)) = c^* e^{rt} \text{ for } t \in [0, t^*],\ q^*(t) = 0 \text{ for } t \in (t^*, T], \tag{3}$$

*and*

$$\int_0^{t^*} q^*(t)dt = S. \tag{4}$$

(b) *The triple $(q^*, c^*, t^*)$ in (a) is uniquely determined by (3) and (4).*

It can be seen from this proposition, in particular from part (b), that, if $(q^*, c^*, t^*)$ is the optimal solution for a given $T$ and if either $t^* < T$, or $t^* = T$ and $q^*(t^*) = 0$, then $(q^*, c^*, t^*)$ is also the optimal solution for any $T'$ with $T' \geqslant t^*$.

We now denote the optimal triple for $T$ by $(q^T, c^T, t^T)$. See [1] for the (simple) proof of the following lemma.

**Lemma 3.3.** *There is a $T > 0$ such that $t^T < T$.*

By Lemma 3.3 and the observation following Proposition 3.2, there is a $\widehat{T} > T^{\mathrm{m}}$ such that $t^T = T$ for all $T \in [T^{\mathrm{m}}, \widehat{T}]$ and $t^T = \widehat{T}$ for all $T \geqslant \widehat{T}$. In view of Lemma 3.1, we may therefore restrict attention to $T \in [T^{\mathrm{m}}, \widehat{T}]$ as the time between two order moments in problem (1).

Let $w^*$ denote the maximum discounted revenue that the seller can receive for selling a single batch of size $S$. Clearly, the selling time that the seller needs to achieve this maximum is at most $\widehat{T}$. On the other hand, it cannot be smaller than $\widehat{T}$ since then the optimal solution of problem (2) for $T = \widehat{T}$ would not be unique, contradicting Proposition 3.2. Thus, we have the following result.

**Lemma 3.4.** $w^* = \int_0^{\widehat{T}} e^{-rt} R(q^{\widehat{T}}(t))dt$.

Our assumption that the seller can make a positive profit on each batch is therefore equivalent to $K < w^*$.

We now turn to the seller's original problem (1). We first restrict our analysis to so-called stationary strategies. A strategy $\sigma =$