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Interchangeability principle and dynamic equations in risk averse stochastic programming

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ABSTRACT

In this paper we consider interchangeability of the minimization operator with monotone risk functionals. In particular we discuss the role of strict monotonicity of the risk functionals. We also discuss implications to solutions of dynamic programming equations of risk averse multistage stochastic programming problems.

1. Introduction

Interchangeability of the minimization and expectation operators is a basis for deriving dynamic programming equations in multistage stochastic programming. In a setting of functional spaces such interchangeability principle is derived, e.g., in Rockafellar and Wets [1, Theorem 14.60]. In a risk averse case interchangeability of the minimization and risk functionals was considered in [2, Theorem 7.1] and [3, Proposition 6.60]. We revisit the question of interchangeability with an emphasis on the role of *strict* monotonicity of considered risk functionals. Importance of such strict monotonicity was already pointed in relation to time consistency of optimal policies of risk averse stochastic programs in [3, Section 6.8.5] and [4]. We also discuss implications of strict monotonicity to solutions of dynamic programming equations.

2. Interchangeability principle

Let $(\Omega, \mathcal{F})$ be a sample space, i.e., $\mathcal{F}$ is a sigma algebra of subsets of Ω, X be an abstract set and $f : X \times \Omega \to \mathbb{R} \cup \{+\infty\}$. Consider

$$F(\omega) := \inf_{x \in X} f(x, \omega). \tag{2.1}$$

Let $\mathcal{Z}$ be a linear space of $\mathcal{F}$-measurable functions $Z : \Omega \to \mathbb{R}$. We discuss the interchangeability principle for each of the following cases.

- ($\aleph$1) The set $\Omega = \{\omega_1, \dots, \omega_m\}$ is finite, $\mathcal{F}$ is the sigma algebra of all subsets of Ω and $\mathcal{Z}$ is the space of all functions $Z : \Omega \to \mathbb{R}$. In this case the space $\mathcal{Z}$ is m-dimensional and can be identified with $\mathbb{R}^m$.
- ($\aleph$2) The sample space $(\Omega, \mathcal{F})$ is equipped with some probability measure P and $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$. Equipped with the norm $\|Z\| = \left(\int |Z|^p dP\right)^{1/p}$ for $p \in [1, \infty)$, and $\|Z\| = \text{ess sup}|Z(\omega)|$ for $p = \infty$, this becomes a Banach space.
- ($\aleph$3) The set Ω is a compact metric space, $\mathcal{F}$ is the Borel sigma algebra of Ω, and $\mathcal{Z} := C(\Omega)$ is the space of continuous functions $Z : \Omega \to \mathbb{R}$ equipped with the sup-norm $\|Z\| = \sup_{\omega \in \Omega} |Z(\omega)|$.

Of course the above case ($\aleph$1), of finite set Ω, can be considered as a particular case of setting ($\aleph$3), we write it separately since in that case the analysis simplifies considerably. In case ($\aleph$2) an element Z of the space $\mathcal{Z}$ is a class of p-integrable functions $Z : \Omega \to \mathbb{R}$ which coincide for all $\omega \in \Omega$ accept on a set of P-measure zero. As we shall discuss it later, the case ($\aleph$3) is relevant when the uncertainty set of probability measures is defined by moment constraints. By writing equalities like $F(\cdot) := \inf_{x \in X} f(x, \cdot)$ we mean that this equality holds for all $\omega \in \Omega$ in cases ($\aleph$1) and ($\aleph$3), and it holds for P-almost every (a.e.) $\omega \in \Omega$ in case ($\aleph$2).

In the above cases ($\aleph$1)–($\aleph$3) there is a naturally defined order relation between $Z, Z' \in \mathcal{Z}$. We write $Z \succeq Z'$ if $Z(\omega) \geq Z'(\omega)$ for all $\omega \in \Omega$ in cases ($\aleph$1) and ($\aleph$3), and $Z(\omega) \geq Z'(\omega)$ for a.e. $\omega \in \Omega$ in case ($\aleph$2). Consider a functional $\mathcal{R} : \mathcal{Z} \to \mathbb{R}$. It is said that $\mathcal{R}$ is monotone if $Z, Z' \in \mathcal{Z}$ and $Z \succeq Z'$, then $\mathcal{R}(Z) \geq \mathcal{R}(Z')$. It is

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said that $\mathcal{R}$ is *strictly* monotone if $\mathcal{R}$ is monotone and $Z \succeq Z'$ and $Z \neq Z'$ imply that $\mathcal{R}(Z) > \mathcal{R}(Z')$. By saying that $\mathcal{R}$ is continuous we mean that it is continuous with respect to the norm topology of the space $\mathcal{Z}$. Let $\mathcal{X}$ be the set of mappings $\chi : \Omega \to X$ such that $f_\chi \in \mathcal{Z}$, where $f_\chi(\cdot) := f(\chi(\cdot), \cdot)$. We also write $\mathcal{R}(f(\chi(\omega), \omega)$ for $\mathcal{R}(f_\chi)$. Consider the following equation

$$\mathcal{R}(F) = \inf_{\chi \in \mathcal{X}} \mathcal{R}(f_\chi), \tag{2.2}$$

and the implications

$$\bar{\chi}(\cdot) \in \arg\min_{x\in X} f(x, \cdot) \Rightarrow \bar{\chi} \in \arg\min_{\chi\in\mathcal{X}} \mathcal{R}(f_\chi), \tag{2.3}$$

$$\bar{\chi} \in \arg\min_{\chi\in\mathcal{X}} \mathcal{R}(f_\chi) \Rightarrow \bar{\chi}(\cdot) \in \arg\min_{x\in X} f(x, \cdot). \tag{2.4}$$

Proposition 2.1. *Suppose that $F \in \mathcal{Z}$ and $\mathcal{R}$ is monotone. Then the following holds.* (i) *Suppose that the minimum of $f(x, \omega)$ over $x \in X$ is attained for all $\omega \in \Omega$. Then* (2.2) and (2.3) *follow; the implication* (2.4) *also follows if either the set* $\arg\min_{\chi\in\mathcal{X}} \mathcal{R}(f_\chi)$ *is a singleton or $\mathcal{R}$ is strictly monotone.* (ii) *Suppose that $\mathcal{R}(\cdot)$ is continuous at F and there exists a sequence $\chi_k \in \mathcal{X}$ such that f_{χ_k} converges to F. Then* (2.2) and (2.3) *follow; the implication* (2.4) *also follows if $\mathcal{R}$ is strictly monotone.*

Proof. We have that $f_\chi \succeq F$ for any $\chi \in \mathcal{X}$. Hence by monotonicity of $\mathcal{R}$ it follows that $\mathcal{R}(f_\chi) \ge \mathcal{R}(F)$, and thus

$$\inf_{\chi\in\mathcal{X}} \mathcal{R}(f_\chi) \ge \mathcal{R}(F).$$

Conversely, consider the setting of case (i), i.e., there exists

$$\bar{\chi}(\cdot) \in \arg\min_{x\in X} f(x, \cdot). \tag{2.5}$$

Then $F = f_{\bar{\chi}}$ and since $F \in \mathcal{Z}$ it follows that $\bar{\chi} \in \mathcal{X}$, and hence

$$\mathcal{R}(F) = \mathcal{R}(f_{\bar{\chi}}) \ge \inf_{\chi\in\mathcal{X}} \mathcal{R}(f_\chi).$$

Thus (2.2) and the implication (2.3) follow. As it was shown above the minimizer $\bar{\chi}$ belongs to the set $\arg\min_{\chi\in\mathcal{X}} \mathcal{R}(f_\chi)$. If this set is a singleton, then the implication (2.4) follows.

Suppose now that $\mathcal{R}$ is strictly monotone. Let $\hat{\chi} \in \arg\min_{\chi\in\mathcal{X}} \mathcal{R}(f_\chi)$. We have that $\mathcal{R}(F) = \mathcal{R}(f_{\hat{\chi}})$. Also $f_{\hat{\chi}} \succeq F$ and hence by strict monotonicity of $\mathcal{R}$ it follows that $f_{\hat{\chi}} = F$, i.e., $f(\hat{\chi}(\cdot), \cdot) = \inf_{x\in X} f(x, \cdot)$. This proves the implication (2.4). This completes the proof of case (i).

Consider now case (ii). Let $\chi_k \in \mathcal{X}$ be a sequence such that f_{χ_k} converges to F. It follows by continuity of $\mathcal{R}$ that

$$\mathcal{R}(F) = \lim_{k\to\infty} \mathcal{R}(f_{\chi_k}) \ge \inf_{\chi\in\mathcal{X}} \mathcal{R}(f_\chi).$$

Hence (2.2) follows, and (2.3) follows as well. If moreover $\mathcal{R}$ is strictly monotone, then the implication (2.4) follows by the same arguments as in case (i). ■

Let us discuss assumptions of the above proposition. In the setting of case ($\aleph$1) the function F belongs to the space $\mathcal{Z}$ if $F(\omega)$ is finite valued, i.e., for every $\omega \in \Omega$ it follows that $\inf_{x\in X} f(x, \omega) > -\infty$ and there is $\bar{x} \in X$ such that $f(\bar{x}, \omega) < \infty$. Also in that case the space $\mathcal{Z}$ is finite dimensional. Consequently if the functional $\mathcal{R} : \mathcal{Z} \to \mathbb{R}$ is convex, then it is continuous. Existence of the corresponding sequence χ_k holds automatically.

In the setting of case ($\aleph$3) suppose that the set X is a compact metric space and $f(x, \omega)$ is finite valued and continuous on $X \times \Omega$. Then $F(\omega)$ is finite valued and continuous, and hence F belongs to the space $C(\Omega)$. Also in that case $f(x, \omega)$ attains its minimal value for every $\omega \in \Omega$, and hence there is no need for the assumption (ii).

In case ($\aleph$2) we need to verify that $F(\omega)$ is measurable and p-integrable for $p \in [1, \infty)$, and essentially bounded for $p = \infty$. Suppose that $X = \mathbb{R}^n$. It is said that function $f(x, \omega)$ is random lower semicontinuous if its epigraphical mapping is closed valued and measurable, [1, Definition 14.28] (in some publications, in particular in [1], such functions are called normal integrands). If $f(x, \omega)$ is random lower semicontinuous, then $F(\omega)$ is measurable, [1, Theorem 14.37]. The condition of p-integrability can be verified by ad hoc methods. In particular this holds if $F(\omega)$ is essentially bounded. Also if $\mathcal{R} : \mathcal{Z} \to \mathbb{R}$ is convex and monotone, then it is continuous in the norm topology of the space $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$ (cf., [2, Proposition 3.1]).

Proposition 2.2. *In the setting of case ($\aleph$2), suppose that $X = \mathbb{R}^n$, $f(x, \omega)$ is random lower semicontinuous, $F \in \mathcal{Z}$ and $\mathcal{R} : \mathcal{Z} \to \mathbb{R}$ is monotone and continuous at F. Then* (2.2) and (2.3) *hold. If moreover $\mathcal{R}$ is strictly monotone, then* (2.4) *holds as well.*

Proof. By the second part of Proposition 2.1 we only need to verify existence of a sequence $\chi_k \in \mathcal{X}$ such that f_{χ_k} converges to F. Consider $\varepsilon > 0$. By the definition (2.1) of function F, for a.e. $\omega \in \Omega$ there is $\bar{\chi}(\omega) \in X$ such that $f(\bar{\chi}(\omega), \omega) < F(\omega) + \varepsilon$. Moreover $\bar{\chi}$ can be chosen in such a way that $f(\bar{\chi}(\cdot), \cdot)$ is measurable. Indeed, since $f(x, \omega)$ is random lower semicontinuous and hence its epigraphical mapping $\omega \mapsto \operatorname{epi} f(\cdot, \omega) \subset \mathbb{R}^n \times \mathbb{R}$ is closed valued and measurable, it follows by the Castaing representation that there is a countable family of measurable mappings $(\chi^\nu, \alpha^\nu) : \Omega \to \mathbb{R}^n \times \mathbb{R}$, $\nu \in \mathbb{N}$, such that for every $\omega \in \Omega$ the set $\{(\chi^\nu(\omega), \alpha^\nu(\omega))\}$ is dense in $\operatorname{epi} f(\cdot, \omega)$, [1, Theorem 14.5]. Consider sets

$$A^\nu := \{\omega \in \Omega : f(\chi^\nu(\omega), \omega) < F(\omega) + \varepsilon\}.$$

It follows that the sets A^ν are measurable and $\cup_{\nu\in\mathbb{N}} A^\nu = \Omega$. Some of these sets can be empty. Define $\bar{\chi}(\omega)$ in the recursive way: $\bar{\chi}(\omega) := \chi^1(\omega)$ for $\omega \in A^1$, and $\bar{\chi}(\omega) := \chi^\nu(\omega)$ for $\omega \in A^\nu \setminus (\cup_{\iota=1}^{\nu-1} A_\iota)$ for $\nu = 2, \ldots$.

Now let ε_k be a sequence of positive numbers converging to zero and $\chi_k(\omega)$ be measurable mappings such that

$$f(\chi_k(\omega), \omega) < F(\omega) + \varepsilon_k, \quad \omega \in \Omega. \tag{2.6}$$

By the definition of $F(\omega)$ we also have that $f(\chi_k(\omega), \omega) \ge F(\omega)$. Since $F \in \mathcal{Z}$ and hence is p-integrable, it follows from (2.6) that f_{χ_k} is also p-integrable and hence $f_{\chi_k} \in \mathcal{Z}$. It also follows from (2.6) that f_{χ_k} converges to F in the norm topology of $\mathcal{Z}$. ■

As the following examples show the *strict* monotonicity condition is essential to ensure the implication (2.4).

Example 1. Consider the setting of case ($\aleph$1) and let $\mathcal{R}(Z) := \sum_{i=1}^m p_i Z(\omega_i)$, where p_i are nonnegative numbers such that $\sum_{i=1}^m p_i = 1$. The functional $\mathcal{R}$ can be viewed as the expectation operator $\mathcal{R} = \mathbb{E}$ associated with probabilities $p_i \ge 0$. This functional is monotone and continuous. Eq. (2.2) takes here the form

$$\mathbb{E}\left[\inf_{x\in X} f(x, \omega)\right] = \inf_{\chi\in\mathcal{X}} \mathbb{E}[f(\chi(\omega), \omega)]. \tag{2.7}$$

If all $p_i > 0$, $i = 1, \ldots, m$, then $\mathcal{R} = \mathbb{E}$ is strictly monotone and both implications (2.3) and (2.4) follow.

Suppose now that one of the probabilities p_i is zero, say $p_1 = 0$. In that case $\mathbb{E}[f(\chi(\omega), \omega)]$ does not depend on $\chi(\omega_1)$ and hence $\bar{\chi}(\omega_1)$ can be any element of the set X in the left hand side of (2.4), provided that such minimizer $\bar{\chi}$ does exist. Hence there is no guarantee that $\bar{\chi}(\omega_1) \in \arg\min_{x\in X} f(x, \omega_1)$ and the implication (2.4) can be false. Of course here the probability of the event $\{\omega_1\}$ is zero, and the implication (2.4) becomes correct if the right hand side of (2.4) is understood to hold w.p.1. In the setting of case ($\aleph$2) the expectation operator is strictly monotone. □