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A greedy algorithm for solving ordinary transportation problem with capacity constraints

Fang Liu

Nanyang Technological University, Singapore

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ABSTRACT

Consider the ordinary transportation problem with the objective to minimize the cost of transporting a single commodity from M warehouses to N demand locations. Each warehouse i has a finite capacity k_i. We convert the above problem into a dual problem and construct a greedy algorithm to solve it.

1. Introduction

Consider ordinary transportation problem for a single commodity with M warehouses and N demand locations. Denote $\mathcal{M} = \{1, \ldots, M\}$ as the set of all warehouses and $\mathcal{N} = \{1, \ldots, N\}$ as the set of all demand locations. At each warehouse $i \in \mathcal{M}$ the maximum amount of product available is $k_i \geq 0$ and at each demand location $j \in \mathcal{N}$ the demand is $d_j \geq 0$. The cost of transporting one unit of the product from warehouse i to demand location j is $c_{ij} \geq 0$. The transportation problem can be formulated into a standard linear programming problem as follows:

$$\begin{aligned} \min_{x_{ij}} \quad & \sum_{i=1}^{M}\sum_{j=1}^{N} c_{ij}x_{ij} \\ s.t. \quad & \sum_{j=1}^{N} x_{ij} \leq k_i \quad \forall i \in \mathcal{M}, \\ & \sum_{i=1}^{M} x_{ij} \geq d_j \quad \forall j \in \mathcal{N}, \\ & x_{ij} \geq 0, \forall i \in \mathcal{M}, j \in \mathcal{N}. \end{aligned} \tag{1}$$

The primal transportation problem (1) has the following dual format

$$\begin{aligned} \max_{u_i, v_j} \quad & \sum_{j=1}^{N} d_j v_j - \sum_{i=1}^{M} k_i u_i \\ s.t. \quad & v_j - u_i \leq c_{ij}, \forall i \in \mathcal{M}, j \in \mathcal{N} \\ & v_j, u_i \geq 0, \forall i \in \mathcal{M}, j \in \mathcal{N}. \end{aligned} \tag{2}$$

Many works have studied conditions under which transportation problems with no capacity constraints can be solved by greedy algorithms (see for example Queyranne et al. [2]). One approach to show greedy algorithms lead to optimality is to verify that the transportation problems satisfy Monge conditions (Hoffman [9], Bein et al. [3]) and the other approach is to consider its dual (Lovász [11], Fujishige and Tomizawa [8], and Edmonds [5]). In Faigle and Fujishige [6], they observe that *"a class of greedy algorithm solvable problems have optimal solutions that have the structure of chains"*. When warehouses have finite capacities, in general greedy algorithm does not lead to optimal solutions and simplex method is not efficient. As a result, many works focus on finding other efficient algorithms to reduce the computational time (Charnes and Cooper [4], Arsham and Kahn [1], Ji and Chu [10]). Within this stream of literature, the most commonly applied algorithm is the stepping stone algorithm that finds the optimal solution by iteratively updating the transportation schedule with feasible cycles of positive cost reductions (Charnes and Cooper [4]). Also, Ji and Chu [10] developed a linear programming approach to compute the optimal solution using the dual formulation. In our paper, we show that the dual problem can be solved by a greedy algorithm. However, different from the greedy algorithm defined in Federgruen and Groenevelt [7] where the increase direction in each iteration is a coordinate direction with the largest increment, in this paper the increase direction is a set of coordinates that results in the largest increment if each coordinate in the set is increased by the same amount.

2. The greedy algorithm

We first convert the dual problem to a concave maximization problem. Note that for any feasible u_i and v_j, we have $v_j \leq c_{ij} + u_i$.

E-mail address: liu_fang@ntu.edu.

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Because $d_j \geq 0$ the optimal v_j^* must satisfy $v_j^* = \min_{i\in\mathcal{M}}\left\{c_{ij}+u_i\right\}$ for all $j \in \mathcal{N}$. Substitute $v_j^* = \min_{i\in\mathcal{M}}\left\{c_{ij}+u_i\right\}$ into Eq. (2), we convert a linear programming problem into a concave optimization problem:

$$\max_{u_i\geq 0, i\in\mathcal{M}} -\sum_{i=1}^{M} k_i u_i + \sum_{j=1}^{N} d_j \min_{i\in\mathcal{M}}\left\{c_{ij}+u_i\right\}. \quad (3)$$

Denote $\mathbf{u} = (u_1, \ldots, u_M)$ and the objective function of (3) as $f(\mathbf{u}) = -\sum_{i=1}^{M} k_i u_i + \sum_{j=1}^{N} d_j \min_{i\in\mathcal{M}}\left\{c_{ij}+u_i\right\}$. We have the following lemma that characterizes properties of the objective function $f(\mathbf{u})$.

Lemma 1. *$f(\mathbf{u})$ is jointly concave and supermodular in $\mathbf{u}$.*

Lemma 1 shows basic properties of $f(\mathbf{u})$. However, these properties are not enough to ensure that the greedy algorithm defined in Federgruen and Groenevelt [7], i.e., increase u_i that leads to the largest increment, is optimal. To see this, consider the following example.

Example 1. Let

$$\begin{aligned} f(u_1,u_2,u_3,u_4) = & -2u_1 - 3u_2 - 2u_3 - 7u_4 \\ & + \min\{u_1, u_2+1, u_3+1, u_4+3\} \\ & +10\min\{u_1, u_2, u_3, u_4+1\} \\ & + \min\{u_1+1, u_2+1, u_3, u_4+3\}. \end{aligned}$$

At $(u_1,u_2,u_3,u_4) = (0,0,0,0)$, the marginal increment of $f(\mathbf{u})$ in the directions u_1, u_2 u_3 and u_4 are all negative, implying that none of the direction u_i will lead to a positive increment in the objective function. However, in the direction $(1,1,1,0)$ the increment of $f(\mathbf{u})$ is 5. Thus, simply increasing the u_i with the largest increment is not optimal.

Example 1 implies that a greedy algorithm only increasing in one coordinate direction at a time may not be optimal. Therefore, we generalize the single-coordinate-greedy-algorithm to include all subsets of coordinate directions. Denote $2^{\mathcal{M}}$ as the class of all subsets of $\mathcal{M}$ and $2^{\mathcal{N}}$ as the class of all subsets of $\mathcal{N}$. Given a feasible $\mathbf{u}$, we define a mapping $\sigma_{\mathbf{u}} : 2^{\mathcal{M}} \longmapsto 2^{\mathcal{N}}$ as $\sigma_{\mathbf{u}}(\mathcal{A}) = \left\{j \middle| \arg\min_i \left\{c_{ij}+u_i\right\} \subseteq \mathcal{A}, j\in\mathcal{N}\right\}$. In other words, $\sigma_{\mathbf{u}}(\mathcal{A})$ is the set of j's that $\min_i\left\{c_{ij}+u_i\right\}$ will increase if we increase all u_i's,$i \in \mathcal{A}$ by the same relatively small amount. Let $\mathbf{e}_{\mathcal{A}}$ be an M dimension vector with the ith element equals 1 if $i\in\mathcal{A}$, and zero otherwise. For any $\mathcal{A} \in 2^{\mathcal{M}}$, define $\Delta_{\mathbf{u}}(\mathcal{A}) = -\sum_{i\in\mathcal{A}} k_i + \sum_{j\in\sigma_{\mathbf{u}}(\mathcal{A})} d_j$ as the increment of $f(\mathbf{u})$ if the increasing direction is $\mathbf{e}_{\mathcal{A}}$. Let $\mathbf{u}_{\mathcal{A}}$ denote an M dimension vector in which the ith element equals u_i if $i\in\mathcal{A}$ and zero otherwise. For two vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u},\mathbf{v}\in\mathcal{M}$, $\mathbf{u}\leq\mathbf{v}$ iff $u_i \leq v_i$, $i\in\mathcal{M}$. Similarly, we can define "$\geq$" and "$=$" for vectors. We have the following lemma that characterizes properties of $\sigma_{\mathbf{u}}(\mathcal{A})$ and $\Delta_{\mathbf{u}}(\mathcal{A})$.

Lemma 2. (1) *$\sigma_{\mathbf{u}}(\emptyset) = \emptyset$. $\sigma_{\mathbf{u}}(\mathcal{A})$ is monotone, that is, for any two sets $\mathcal{A}\subseteq\mathcal{B}\in 2^{\mathcal{M}}$, we have $\sigma_{\mathbf{u}}(\mathcal{A}) \subseteq \sigma_{\mathbf{u}}(\mathcal{B})$.*

(2) *$\sigma_{\mathbf{u}}(\mathcal{A})$ is superadditive in union, that is, for any two sets $\mathcal{A}$, $\mathcal{B}\in 2^{\mathcal{M}}$, we have $\sigma_{\mathbf{u}}(\mathcal{A}) \cup \sigma_{\mathbf{u}}(\mathcal{B}) \subseteq \sigma_{\mathbf{u}}(\mathcal{A}\cup\mathcal{B})$. Therefore, $\sigma_{\mathbf{u}}(\mathcal{A})$ is also supermodular.*

(3) *$\sigma_{\mathbf{u}}(\mathcal{A})$ is additive in interception, that is, for any two sets $\mathcal{A}$, $\mathcal{B}\in 2^{\mathcal{M}}$, we have $\sigma_{\mathbf{u}}(\mathcal{A}) \cap \sigma_{\mathbf{u}}(\mathcal{B}) = \sigma_{\mathbf{u}}(\mathcal{A}\cap\mathcal{B})$.*

(4) *$\Delta_{\mathbf{u}}(\emptyset) = 0$. $\Delta_{\mathbf{u}}(\mathcal{A})$ is supermodular, that is, for any two sets $\mathcal{A},\mathcal{B}\in 2^{\mathcal{M}}$ we have $\Delta_{\mathbf{u}}(\mathcal{A}) + \Delta_{\mathbf{u}}(\mathcal{B}) \leq \Delta_{\mathbf{u}}(\mathcal{A}\cup\mathcal{B}) + \Delta_{\mathbf{u}}(\mathcal{A}\cap\mathcal{B})$.*

(5) For any two sets $\mathcal{A}$, $\mathcal{B} \in 2^{\mathcal{M}}$ *and* $\mathcal{A}\cap\mathcal{B} = \emptyset$ *, we have* $\Delta_{\mathbf{u}}(\mathcal{A}) + \Delta_{\mathbf{u}}(\mathcal{B}) \leq \Delta_{\mathbf{u}}(\mathcal{A}\cup\mathcal{B})$.

(6) For any $\mathbf{u}^1$ *and* $\mathbf{u}^2$, *if* $\mathbf{u}^1_{\mathcal{A}} = \mathbf{u}^2_{\mathcal{A}}$ *and* $\mathbf{u}^1_{\mathcal{M}\setminus\mathcal{A}} \leq \mathbf{u}^2_{\mathcal{M}\setminus\mathcal{A}}$, *then* $\sigma_{\mathbf{u}^1}(\mathcal{A}) \subseteq \sigma_{\mathbf{u}^2}(\mathcal{A})$ *and* $\Delta_{\mathbf{u}^1}(\mathcal{A}) \leq \Delta_{\mathbf{u}^2}(\mathcal{A})$.

We define the greedy algorithm as follows.

Algorithm 1. (1) Initialize: $\mathbf{u} = \mathbf{0}$, $\mathcal{A}_{max} \in \arg\max_{\mathcal{A}\in 2^{\mathcal{M}}} \Delta_{\mathbf{u}}(\mathcal{A})$;
(2) While $\Delta_{\mathbf{u}}(\mathcal{A}_{max}) > 0$, do
$\mathbf{u} = \mathbf{u} + \sup\left\{u \middle| \sigma_{\mathbf{u}} = \sigma_{\mathbf{u}+u\mathbf{e}_{\mathcal{A}_{max}}}\right\}\mathbf{e}_{\mathcal{A}_{max}}$, $\mathcal{A}_{max} \in \arg\max_{\mathcal{A}\in 2^{\mathcal{M}}} \Delta_{\mathbf{u}}(\mathcal{A})$;
return $\mathbf{u}^* = \mathbf{u}$.

Next, we show that $\mathbf{u}^*$, the solution returned by Algorithm 1, is always optimal. Since $f(\mathbf{u})$ is concave in $\mathbf{u}$, it is sufficient to show that $\mathbf{u}^*$ is a local maximum. Similar to $\sigma_{\mathbf{u}}$, we define a mapping $\pi_{\mathbf{u}} : 2^{\mathcal{M}} \longmapsto 2^{\mathcal{N}}$ as $\pi_{\mathbf{u}}(\mathcal{A}) = \left\{j \middle| \arg\min_i\left\{c_{ij}+u_i\right\} \cap \mathcal{A} \neq \phi, j\in\mathcal{N}\right\}$. In other words, $\sigma_{\mathbf{u}}(\mathcal{A})$ is the set of js that $\min_i\left\{c_{ij}+u_i\right\}$ will decrease if we decrease all u_is,$i\in\mathcal{A}$ for a same significantly small amount. For any $\mathcal{A}\in 2^{\mathcal{M}}$, define $\Lambda_{\mathbf{u}}(\mathcal{A}) = \sum_{i\in\mathcal{A}} k_i - \sum_{j\in\pi_{\mathbf{u}}(\mathcal{A})} d_j$ as the decrement of $f(\mathbf{u})$ if the decreasing direction is $\mathcal{A}$. We have the following lemma that characterizes properties of $\pi_{\mathbf{u}}(\mathcal{A})$ and $\Lambda_{\mathbf{u}}(\mathcal{A})$.

Lemma 3. (1) *$\pi_{\mathbf{u}}(\emptyset) = \emptyset$. $\pi_{\mathbf{u}}(\mathcal{A})$ is monotone, that is, for any two sets $\mathcal{A}\subseteq\mathcal{B}\in 2^{\mathcal{M}}$, we have $\pi_{\mathbf{u}}(\mathcal{A}) \subseteq \pi_{\mathbf{u}}(\mathcal{B})$.*

(2) *$\pi_{\mathbf{u}}(\mathcal{A})$ is additive in union, that is, for any two sets $\mathcal{A},\mathcal{B}\in 2^{\mathcal{M}}$, we have $\pi_{\mathbf{u}}(\mathcal{A}) \cup \pi_{\mathbf{u}}(\mathcal{B}) = \pi_{\mathbf{u}}(\mathcal{A}\cup\mathcal{B})$.*

(3) *$\pi_{\mathbf{u}}(\mathcal{A})$ is subadditive in interception, that is, for any two sets $\mathcal{A},\mathcal{B}\in 2^{\mathcal{M}}$, we have $\pi_{\mathbf{u}}(\mathcal{A}) \cap \pi_{\mathbf{u}}(\mathcal{B}) \supseteq \pi_{\mathbf{u}}(\mathcal{A}\cap\mathcal{B})$.*

(4) *$\Lambda_{\mathbf{u}}(\phi) = 0$. $\Lambda_{\mathbf{u}}(\mathcal{A})$ is supermodular, that is, for any two sets $\mathcal{A},\mathcal{B}\in 2^{\mathcal{M}}$ we have $\Lambda_{\mathbf{u}}(\mathcal{A}) + \Lambda_{\mathbf{u}}(\mathcal{B}) \leq \Lambda_{\mathbf{u}}(\mathcal{A}\cup\mathcal{B}) + \Lambda_{\mathbf{u}}(\mathcal{A}\cap\mathcal{B})$.*

(5) For any $\mathbf{u}^1$ *and* $\mathbf{u}^2$, *if* $\mathbf{u}^1_{\mathcal{A}} \leq \mathbf{u}^2_{\mathcal{A}}$ *and* $\mathbf{u}^1_{\mathcal{M}\setminus\mathcal{A}} = \mathbf{u}^2_{\mathcal{M}\setminus\mathcal{A}}$, *then* $\pi_{\mathbf{u}^1}(\mathcal{A}) \supseteq \pi_{\mathbf{u}^2}(\mathcal{A})$ *and* $\Lambda_{\mathbf{u}^1}(\mathcal{A}) \leq \Lambda_{\mathbf{u}^2}(\mathcal{A})$.

(6) *For any $\mathbf{u}^1$ and $\mathbf{u}^2$, if $\mathbf{u}^1_{\mathcal{A}} = \mathbf{u}^2_{\mathcal{A}}$ and $\mathbf{u}^1_{\mathcal{M}\setminus\mathcal{A}} \leq \mathbf{u}^2_{\mathcal{M}\setminus\mathcal{A}}$, then $\pi_{\mathbf{u}^1}(\mathcal{A}) \subseteq \pi_{\mathbf{u}^2}(\mathcal{A})$ and $\Lambda_{\mathbf{u}^1}(\mathcal{A}) \geq \Lambda_{\mathbf{u}^2}(\mathcal{A})$.*

(7) *For all $\mathcal{A}\in 2^{\mathcal{M}}$, $\pi_{\mathbf{u}}(\mathcal{A}) \supseteq \sigma_{\mathbf{u}}(\mathcal{A})$ and $\Lambda_{\mathbf{u}}(\mathcal{A}) + \Delta_{\mathbf{u}}(\mathcal{A}) \leq 0$.*

The following theorem shows that Algorithm 1 returns an optimal solution of the dual problem.

Theorem 1. *If Algorithm 1 stops at some $\mathbf{u}$, then it returns the optimal policy of the dual problem, $\mathbf{u} = \mathbf{u}^*$. Otherwise, the dual problem is unbounded and the primal problem is infeasible. In each iteration, $\mathcal{A}_{max}$ can be found by maximizing a supermodular set function. Thus, (2) can be solved by a greedy algorithm.*

Proof. First, it is clear that when the algorithm stops at some $\mathbf{u}$, the objective function cannot increase in any direction by adding a positive amount to $\mathbf{u}$, that is, $\Delta_{\mathbf{u}}(\mathcal{A}) \leq 0$ for all $\mathcal{A} \subseteq 2^{\mathcal{M}}$. To show Algorithm 1 is optimal, we only need to show that $\mathbf{u}$ is a local maximum, which is equivalent to verifying that the algorithm stops at some $\mathbf{u} = \mathbf{u}^*$ such that $\Lambda_{\mathbf{u}^*}(\mathcal{A}) \leq 0$ for all $\mathcal{A}\subseteq 2^{\mathcal{M}}$.

Now suppose the above is not true, that is, there exists a set $\mathcal{R}$ such that $\Lambda_{\mathbf{u}^*}(\mathcal{R}) > 0$. Denote the set $\mathcal{A}^*_{max} \subseteq 2^{\mathcal{M}}$ as the set that is picked for increment before reaching $\mathbf{u}^*$. If $\mathcal{R}\subseteq\mathcal{A}^*_{max}$, then for an arbitrary small $\epsilon > 0$, from the definitions of $\Delta_{\mathbf{u}}(\mathcal{A})$ and $\Gamma_{\mathbf{u}}(\mathcal{A})$, we have

$$\begin{aligned} \Delta_{\mathbf{u}^*-\epsilon\mathbf{u}_{\mathcal{A}^*_{max}}}\left(\mathcal{A}^*_{max}\setminus\mathcal{R}\right) &= \Delta_{\mathbf{u}^*-\epsilon\mathbf{u}_{\mathcal{A}^*_{max}}}\left(\mathcal{A}^*_{max}\right) + \Lambda_{\mathbf{u}^*-\epsilon\mathbf{u}_{\mathcal{A}^*_{max}}}(\mathcal{R}) \\ &> \Delta_{\mathbf{u}^*-\epsilon\mathbf{u}_{\mathcal{A}^*_{max}}}\left(\mathcal{A}^*_{max}\right), \end{aligned}$$

which contradicts with $\mathcal{A}^*_{max} \in \arg\max_{\mathcal{A}\in 2^{\mathcal{M}}} \Delta_{\mathbf{u}^*-\epsilon\mathbf{u}_{\mathcal{A}^*_{max}}}(\mathcal{A})$.

Therefore, we can safely assume $\mathcal{R} \nsubseteq \mathcal{A}^*_{max}$. Then consider the last iteration that some elements in $\mathcal{R}$ are selected in Algorithm 1 for increment. That is, all iterations afterwards only increases u_is that $i\notin\mathcal{R}$. Suppose $\mathbf{u}^0$ is the vector at the beginning of that iteration, $\mathbf{u}^1$ is the vector at the end of the iteration and $\mathcal{A}^1_{max}$ is the corresponding increment set. Then according to Lemma 2(6) part two,

$$\Delta_{\mathbf{u}^1}\left(\mathcal{A}^*_{max}\right) \geq \Delta_{\mathbf{u}^*}\left(\mathcal{A}^*_{max}\right) > 0.$$