Operations Research Letters 45 (2017) 392–396

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

The maximum distribution of Kibble's bivariate gamma random vector

Bara Kim [a], Jeongsim Kim [b,*]

[a] Department of Mathematics, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul, 02841, Republic of Korea
[b] Department of Mathematics Education, Chungbuk National University, 1 Chungdae-ro, Seowon-gu, Cheongju, Chungbuk, 28644, Republic of Korea

ARTICLE INFO

Article history:
Received 13 July 2015
Received in revised form 8 June 2017
Accepted 8 June 2017
Available online 19 June 2017

Keywords:
Kibble's bivariate gamma distribution
Downton's bivariate exponential distribution
First passage time

ABSTRACT

Bivariate gamma distribution (BGD) can be used in hydrology, stochastic modeling and reliability theory. We derive the Laplace–Stieltjes transform of the distribution of $\max\{Y_1, Y_2\}$ when a random vector (Y_1, Y_2) follows Kibble's BGD with integral shape parameter. This is achieved by showing that $\max\{Y_1, Y_2\}$ has the same distribution as the first passage time of a continuous time Markov process.

1. Introduction

Bivariate gamma distributions (BGDs) can be used to construct models for positively correlated data, with positive skewness in each dimension. Unlike the normal distribution, the gamma distribution does not have a unique natural extension to the bivariate or the multivariate case. Therefore, a number of different kinds of BGDs have been proposed, refer to Balakrishnan and Lai [3]. One particular BGD that has received considerable attention is the one proposed by Kibble [9].

Kibble's BGD requires four parameters $\alpha > 0$, $\mu_1 > 0$, $\mu_2 > 0$ and ρ, $0 \le \rho < 1$. If a random vector (Y_1, Y_2) follows Kibble's BGD with parameters α, μ_1, μ_2 and ρ, then the joint probability density function is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{(\mu_1\mu_2)^\alpha}{(1-\rho)\Gamma(\alpha)}\left(\frac{y_1y_2}{\rho\mu_1\mu_2}\right)^{\frac{\alpha-1}{2}} \times \exp\left(-\frac{\mu_1y_1+\mu_2y_2}{1-\rho}\right) \times I_{\alpha-1}\left(\frac{2\sqrt{\rho\mu_1\mu_2y_1y_2}}{1-\rho}\right), \quad (1)$$

for $y_1 > 0, y_2 > 0$, where $\Gamma(\alpha)$ is the gamma function defined as $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$, and $I_\alpha(\cdot)$ is the modified Bessel function of the first kind of order α defined as

$$I_\alpha(z) = \sum_{j=0}^{\infty} \frac{1}{\Gamma(\alpha+j+1)j!}\left(\frac{z}{2}\right)^{2j+\alpha}.$$

The marginal distributions of Y_1 and Y_2 are both gamma with the same shape parameter α, and rate parameters μ_1 and μ_2, respectively, and the correlation coefficient between Y_1 and Y_2 is ρ. The joint Laplace–Stieltjes transform (LST) is given by

$$\mathbb{E}[e^{-s_1Y_1-s_2Y_2}] = \left(\frac{\mu_1\mu_2}{(\mu_1+s_1)(\mu_2+s_2)-\rho s_1s_2}\right)^\alpha.$$

In the special case when $\alpha = 1$, Kibble's BGD is the well-known Downton's bivariate exponential distribution (BED) with three parameters μ_1, μ_2, and ρ (refer to [6]).

Kibble's BGD has been used in several research areas. For example, Phatarford [13] used this distribution as a model to describe summer and winter streamflows. Izawa [8] used this distribution to describe the joint distribution of rainfall at two nearby rain gauges. Smith et al. [14] investigated applications to wind gust modeling for the ascent flight of the Space Shuttle. Chatelain et al. [4] studied applications to image registration and change detection. For applications of Downton's BED to queueing systems, refer to Conolly and Choo [5], Kim et al. [10] and Langaris [11].

Iliopoulos et al. [7] described Bayesian estimation for the parameters of Kibble's BGD through a Markov chain Monte Carlo scheme. Izawa [8] obtained the density functions and moments of Y_1+Y_2, Y_1Y_2, and Y_1/Y_2, when a random vector (Y_1, Y_2) follows Kibble's BGD with the same rate parameters (i.e., $\mu_1 = \mu_2$). Nadarajah and Kotz [12] derived the density functions and moments of Y_1+Y_2

* Corresponding author.
E-mail addresses: bara@korea.ac.kr (B. Kim), jeongsimkim@chungbuk.ac.kr (J. Kim).

http://dx.doi.org/10.1016/j.orl.2017.06.001
0167-6377/

and $Y_1/(Y_1+Y_2)$, when $\mu_1=\mu_2$. Whereas the distributions of extreme statistics $\min\{Y_1,Y_2\}$ and $\max\{Y_1,Y_2\}$ were obtained only when a random vector (Y_1,Y_2) follows BED, refer to Downton [6].

In this paper, we derive the LST of the distribution of $\max\{Y_1,Y_2\}$ when (Y_1,Y_2) has Kibble's BGD with integral shape parameter α. This is achieved by showing that $\max\{Y_1,Y_2\}$ has the same distribution as the first passage time of a continuous time Markov process, and then by analyzing the first passage time by the matrix analytic method. This paper generalizes the result of Downton [6], where he obtained the LST of the distribution of $\max\{Y_1,Y_2\}$ when $\alpha=1$, by direct integration of the joint probability density function (1). The generalization is nontrivial because the method of direct integration used by Downton [6] is difficult to be applied to the case of general α. It is noticeable that the method used in our generalization is completely different from that of Downton [6].

2. A representation of Kibble's bivariate gamma random vector

Let a random vector (Y_1,Y_2) have Kibble's BGD with parameters α,μ_1,μ_2, and ρ, where $\alpha=n$ for an integer n. In this section we will show that the distribution of $\max\{Y_1,Y_2\}$ is represented as the first passage time of a continuous time Markov process. For notational convenience, we set

$$\nu_i=\frac{\mu_i}{1-\rho},\ \gamma_i=\frac{\rho\mu_i}{1-\rho},\quad i=1,2,$$

and so $\mu_i=(1-\rho)\nu_i$, $\gamma_i=\rho\nu_i$, $\mu_i+\gamma_i=\nu_i$, $i=1,2$. We consider a two-dimensional continuous time Markov process $\{(M(t),J(t)):t\geq 0\}$ with the state space $\{(n,i):n=0,\pm1,\pm2,\ldots,i=0,1,2,\ldots\}$, and the state transition diagram shown in Fig. 1. For $l=\pm1,\pm2,\ldots,i=0,1,\ldots$, let $\tau_{(0,0)}^{(l,i)}$ be the first passage time from state (l,i) to state $(0,0)$ by the process $\{(M(t),J(t)):t\geq 0\}$. As shown in the following theorem, $\max\{Y_1,Y_2\}$ has the same distribution as the first passage time from state $(0,n)$ to state $(0,0)$ by the process $\{(M(t),J(t)):t\geq 0\}$. From now on, "$\stackrel{d}{=}$" denotes "equal in distribution".

Theorem 1. *If (Y_1,Y_2) follows Kibble's BGD with parameters $\alpha=n$, μ_1,μ_2, and ρ, then*

$$\max\{Y_1,Y_2\}\stackrel{d}{=}\tau_{(0,0)}^{(0,n)}.$$

We need the following three lemmas to prove Theorem 1. To present the first lemma, we note that Kibble's bivariate random vector is expressed as the sum of Downton's bivariate random vectors. Let $\{(X_{n1},X_{n2}):n=1,2,\ldots\}$ be a sequence of independent and identically distributed (i.i.d.) bivariate random vectors with Downton's BED of parameters μ_1,μ_2 and ρ, and put

$$S_{ni}=\sum_{k=1}^{n}X_{ki},\quad i=1,2.$$

Then (S_{n1},S_{n2}) has Kibble's BGD with parameters $\alpha=n,\mu_1,\mu_2$, and ρ. The following lemma provides a simple representation for (S_{n1},S_{n2}), which follows from the result of Al-saadi and Young [2].

Lemma 1. *For $i=1,2$, let $\{\mathcal{E}_{\nu_i}^{(k,i)}:k=1,2,\ldots\}$ be a sequence of i.i.d. random variables with an exponential distribution of parameter ν_i. Let $\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_n$ be i.i.d. random variables with a geometric distribution of parameter $1-\rho$, i.e., $\mathbb{P}(\mathcal{N}_i=k)=\rho^{k-1}(1-\rho)$, $k=1,2,\ldots$. Assume that the random variables $\mathcal{E}_{\nu_i}^{(k,i)}$, $k=1,2,\ldots$, $i=1,2$ and $\mathcal{N}_i$, $i=1,\ldots,n$ are independent. Then we have*

$$(S_{n1},S_{n2})\stackrel{d}{=}\left(\sum_{k=1}^{N_n}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_n}\mathcal{E}_{\nu_2}^{(k,2)}\right),\quad n=1,2,\ldots,$$

where $N_n=\sum_{i=1}^{n}\mathcal{N}_i$ is a negative binomial random variable with parameters n and $1-\rho$.

To state the next lemma, we introduce the random vector

$$(I_1(p_1,p_2,\ldots,p_n),I_2(p_1,p_2,\ldots,p_n),\ldots,I_n(p_1,p_2,\ldots,p_n)),$$

which has a multinomial distribution with parameters 1 and $(p_1,\ldots,p_n)$. For example, $(I_1(p_1,p_2,p_3),I_2(p_1,p_2,p_3),I_3(p_1,p_2,p_3))$ has value $(1,0,0)$ with probability p_1, value $(0,1,0)$ with probability p_2, and value $(0,0,1)$ with probability p_3. With this notation we have the following lemma. The proof is given in the online supplementary material (see Appendix A).

Lemma 2. *Let $\{\mathcal{E}_{\nu_i}^{(k,i)}:k=1,2,\ldots\}$, $i=1,2$ and N_n be the same as in Lemma 1. Moreover, let $\mathcal{E}_\nu$ be a random variable with an exponential distribution of parameter ν. Assume that all these random variables are independent. Then we have the following.*

(i) *For $n=1,2\ldots$, $m=1,2\ldots$,*

$$\begin{aligned}\max\Big\{\sum_{k=1}^{N_n+m}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_n}\mathcal{E}_{\nu_2}^{(k,2)}\Big\}
&\stackrel{d}{=}\mathcal{E}_{\mu_2+\gamma_2+\nu_1}\\
&\quad+I_1(q_1,q_2,q_3)\max\Big\{\sum_{k=1}^{N_{n-1}+m+1}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_{n-1}}\mathcal{E}_{\nu_2}^{(k,2)}\Big\}\\
&\quad+I_2(q_1,q_2,q_3)\max\Big\{\sum_{k=1}^{N_n+m+1}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_n}\mathcal{E}_{\nu_2}^{(k,2)}\Big\}\\
&\quad+I_3(q_1,q_2,q_3)\max\Big\{\sum_{k=1}^{N_n+m-1}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_n}\mathcal{E}_{\nu_2}^{(k,2)}\Big\},\end{aligned}\tag{2}$$

where $(q_1,q_2,q_3)=\left(\frac{\mu_2}{\mu_2+\gamma_2+\nu_1},\frac{\gamma_2}{\mu_2+\gamma_2+\nu_1},\frac{\nu_1}{\mu_2+\gamma_2+\nu_1}\right)$ and $I_i(q_1,q_2,q_3)$, $i=1,2,3$, are independent of all other random variables.

(ii) *For $n=1,2\ldots$, $m=1,2,\ldots$,*

$$\begin{aligned}\max\Big\{\sum_{k=1}^{N_n}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_n+m}\mathcal{E}_{\nu_2}^{(k,2)}\Big\}
&\stackrel{d}{=}\mathcal{E}_{\mu_1+\gamma_1+\nu_2}\\
&\quad+I_1(r_1,r_2,r_3)\max\Big\{\sum_{k=1}^{N_{n-1}}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_{n-1}+m+1}\mathcal{E}_{\nu_2}^{(k,2)}\Big\}\\
&\quad+I_2(r_1,r_2,r_3)\max\Big\{\sum_{k=1}^{N_n}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_n+m+1}\mathcal{E}_{\nu_2}^{(k,2)}\Big\}\\
&\quad+I_3(r_1,r_2,r_3)\max\Big\{\sum_{k=1}^{N_n}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_n+m-1}\mathcal{E}_{\nu_2}^{(k,2)}\Big\},\end{aligned}$$

where $(r_1,r_2,r_3)=\left(\frac{\mu_1}{\mu_1+\nu_2+\gamma_1},\frac{\gamma_1}{\mu_1+\nu_2+\gamma_1},\frac{\nu_2}{\mu_1+\nu_2+\gamma_1}\right)$, and $I_i(r_1,r_2,r_3)$, $i=1,2,3$, are independent of all other random variables.

(iii) *For $n=1,2\ldots$,*

$$\begin{aligned}\max\Big\{\sum_{k=1}^{N_n}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_n}\mathcal{E}_{\nu_2}^{(k,2)}\Big\}
&\stackrel{d}{=}\mathcal{E}_{\mu_1+\mu_2+\gamma_1+\gamma_2}\\
&\quad+I_1(\delta_1,\delta_2,\delta_3,\delta_4)\max\Big\{\sum_{k=1}^{N_{n-1}+1}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_{n-1}}\mathcal{E}_{\nu_2}^{(k,2)}\Big\}\\
&\quad+I_2(\delta_1,\delta_2,\delta_3,\delta_4)\max\Big\{\sum_{k=1}^{N_{n-1}}\mathcal{E}_{\nu_1}^{(k,1)},\sum_{k=1}^{N_{n-1}+1}\mathcal{E}_{\nu_2}^{(k,2)}\Big\}\end{aligned}$$