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Closed-form formulae for moment, tail probability, and blocking probability of waiting time in a buffer-sharing deterministic system

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ABSTRACT

Obtaining analytic expressions for characteristics in probabilistic systems with finite buffer capacities such as (higher) moments and tail probabilities of stationary waiting times, and blocking probabilities is by no means trivial. This is also true even for a system with deterministic processing times. By using the max–plus algebraic approach in this study, we introduce closed-form formulae for characteristics of stationary waiting time in a complete buffer-sharing m-node tandem system with constant processing times. Numerical examples are also provided.

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1. Introduction

For design and maintenance purposes, it is sometimes necessary to investigate the queueing mechanisms embedded in various computer systems, telecommunication networks, and automated manufacturing systems. Although finite-capacity queueing systems have been widely studied, research in this area has rendered few explicit results. Due to the difficulties posed by finite capacities, analytic solutions are difficult to obtain; most studies have been limited in number of nodes, distributions of arrivals, service times, and so on. There are, however, a few analytic solutions to finite-capacity queues for special cases such as $M/M/1/K$, $M/G/c/c$, $M/G/1/1$, $M/G/1/2$, and so on. Tijms [14] presented recursive formulae for stationary distributions and blocking probabilities in $M/G/c/K$ queues (also see Takagi [13]). Brun and Garcia [6] derived analytical (transform-free) solutions for steady-state distribution in $M/D/1/K$ queues by using a generating function. The standard queueing theory is not yet applicable, however, to general queues such as multi-node, multi-server, and generally structured queues. To achieve analytic solutions for multi-node systems, most studies have used decomposition methods by decomposing the network into subnetworks and treating each subnetwork independently with adjusted parameters such as input rates and service rates (see, e.g., Jun and Perros [7]; Shi [12]; and the references therein).

Unlike the infinite buffer case, the distribution of waiting time in tandem queues with a finite buffer is not simply given as a product form due to the blocking phenomena between nodes. Therefore, various approximation methods using decomposition and simulation have been proposed. In this study, however, we develop an exact solution procedure based on max–plus algebra. The max–plus linear system uses only two operators, "max" and "plus", to represent its performance characteristics. It is well known that the max–plus linear system (MPL) includes various probabilistic systems commonly found in telecommunication and computer networks. Ayhan and Seo [1], Baccelli et al. [3] provided some preliminaries on max–plus algebraic representation of waiting times in MPLs.

Conceptually, a buffer sharing policy can be applied to various systems without system configuration limitations. However, we here focus on a tandem system consisting of m nodes having constant processing times and having a Poisson arrival process with rate λ in order to obtain analytical solutions for waiting time perspective.

Two typical blocking policies adopted in many researches are communication blocking (blocking before service) and production blocking (blocking after service). Under a production blocking policy the common buffer is occupied in advance by a blocked job waiting at node 0 (a dummy node). However, under a communication blocking policy the common buffer is occupied only when a blocked job becomes unblocked and is joining in node 1 (the first node in our actual system). Communication blocking is more suitable for representing the blocking phenomena prevalent in general

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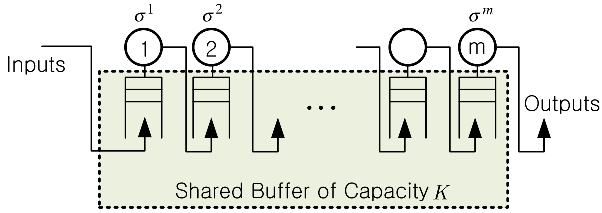

Fig. 1. A buffer-sharing tandem queue.

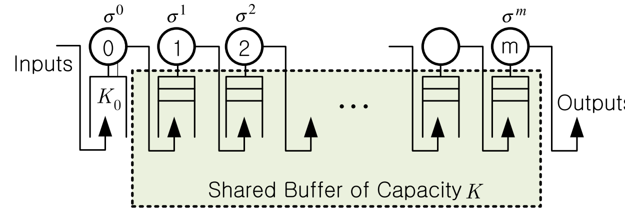

Fig. 2. A buffer-sharing tandem queue with a dummy node.

queueing systems and has simpler expressions in max–plus algebraic notation than production blocking. Thus it is assumed that pulling a job between nodes 0 and 1 in the system follows a communication blocking policy. Under a complete buffer sharing policy, we introduce explicit expressions for higher moments and tail probability of stationary waiting times in an m-node tandem system with constant processing times, and also obtain a closed-form formula for blocking probability.

2. Explicit expression for moments of waiting time

First we introduce brief preliminaries on max–plus algebraic approach. Baccelli and Schmidt [5] introduced that the dynamics of max–plus linear systems with α nodes can be described by the α-dimensional vectorial recurrence equations

$$X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1} \tag{1}$$

with an initial condition of X_0, where the $\oplus$ refers to maximization and the $\otimes$ refers to addition for scalars and max–plus algebra product for matrices, $\{T_n\}$ is a non-decreasing sequence of real-valued random numbers (e.g. the epochs of the Poisson arrival process with rate λ), $\{A_n\}$ and $\{B_n\}$ are stationary and ergodic sequences of real-valued random matrices of size $\alpha \times \alpha$ and $\alpha \times 1$, respectively, and $\{X_n\}$ is a sequence of α-dimensional state vectors referring to the absolute time of the beginning of the nth service at each node. One is more interested in the differences $W_n^i = X_n^i - T_n$ (like the waiting time of the nth customer until he joins server i). Let $\tau_n = T_{n+1} - T_n$ with $T_0 = 0$ and let $C(x)$ be the $\alpha \times \alpha$ matrix with all diagonal entries equal to $-x$ and all non-diagonal entries equal to $-\infty$. By subtracting T_{n+1} from both sides of (1), the new state vector W_{n+1} can be expressed as

$$W_{n+1} = A_n \otimes C(\tau_n) \otimes W_n \oplus B_{n+1},$$

for $n \geq 0$ and with the initial condition W_0. They also demonstrated that under certain conditions, the dynamics of Poisson driven max–plus linear systems could be described by vectorial recurrence equations. For all $\lambda < a^{-1}$, where λ is the arrival rate and a is the maximal Lyapunov exponent of the sequence $\{A_n\}$, a stationary waiting time W is determined by the matrix-series

$$W = D_0 \oplus \bigoplus_{k\geq 1} C(T_{-k}) \otimes D_k$$

with $D_0 = B_0$, $W_0 = B_0$, and for all $k \geq 1$

$$D_k = \left(\bigotimes_{n=1}^{k} A_{-n}\right) \otimes B_{-k}. \tag{2}$$

Note that the random vector D_n plays an important role in computing characteristics of waiting times, and D_n^i, the ith component of D_n, refers to the time elapsed from arrival until the beginning of his process at node i when there exist n customers in the system and they can be interpreted as a critical path in a task graph.

Later, under certain conditions, Baccelli et al. [4] showed that for a stationary Poisson process with intensity λ and a Riemann integrable non-negative function $G : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, the expectation $\mathbb{E}[G(W^i)]$ of the functional $G(\cdot)$ applied to the ith component of the steady-state vector W can be expanded as a Taylor series of order m with respect to λ, i.e.,

$$\mathbb{E}[G(W^i)] = \sum_{k=0}^{m} \lambda^k \mathbb{E}[q_{k+1}(D_0^i, \ldots, D_k^i)] + O(\lambda^{m+1}) \tag{3}$$

for all arrival intensities $\lambda \in [0, a^{-1}]$, where

$$\begin{aligned} &q_{k+1}(x_0, x_1, \ldots, x_k) \\ &= \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} H^{[k]}(x_n) - \sum_{n=0}^{k-1} \sum_{j=n}^{k-1} \binom{j}{n} (-1)^{j-n} H^{[j]}(x_n) \\ &\quad \left\{p_{k-j}(x_{n+1}, \ldots, x_{k-j+n}) - p_{k-j}(x_n, \ldots, x_{k-j+n-1})\right\} \end{aligned}$$

with $H^{[0]}(x) = G(x)$ and $H^{[n]}(x)$ is recursively defined by a suitably chosen version of the indefinite Riemann-integral $\int H^{[n-1]}(x)dx$. Baccelli and Schmidt [5] first defined the polynomials $p_k(\cdots)$ in a different way, but it is shown in Baccelli et al. [4] that when $G(x) = x$ the polynomials $q_k(\cdots)$ give alternative expressions of the polynomials $p_k(\cdots)$ (see (7)).

A buffer-sharing system consisting of m nodes in series is shown in Fig. 1. Let σ^i, $i = 1, \ldots, m$, be a processing time at node i and $K(\geq m)$ be the capacity of a common buffer completely shared by m nodes. We assume that arrivals from the outside follow a Poisson process with rate λ. A stability condition ($\rho < 1$) and an unlimited buffer capacity of the first node are basic assumptions in our max–plus algebraic approach, the Taylor series expansion. Thus, the finite-capacity assumption of the first node depicted in Fig. 1 needs to be relaxed. The relaxed model can be obtained by inserting a dummy node (node 0) with zero processing time ($\sigma^0 = 0$) and infinite capacity ($K_0 = \infty$) as shown in Fig. 2. This dummy node is assumed to describe unlimited arrivals of jobs to systems that never starve. Figure 2 shows a line production system consisting of m nodes where they share a common buffer of capacity $K(\geq m)$.

To avoid unnecessary complex notation we reduce the number of nodes by one and renumber them from 1 to m. Then we can obtain the following recurrence expression for X_n^i, the time epoch which the nth process can start at node i. For node 1,

$$X_{n+1}^1 = \sigma_{n+1}^0 \otimes X_{n+1}^0 \oplus \sigma_n^1 \otimes X_n^1 \oplus \sigma_{n-K+1}^m \otimes X_{n-K+1}^m, \tag{4}$$

and for node i, $i \geq 2$,

$$X_{n+1}^i = \sigma_{n+1}^{i-1} \otimes X_{n+1}^{i-1} \oplus \sigma_n^i \otimes X_n^i \tag{5}$$

where σ_n^i is the nth processing time at node i for $i \geq 1$ and $\sigma_n^0 (= \tau_n)$ is the interarrival time between the nth arrival and his predecessor in the external arrival stream. Because these expressions satisfy the expression of (1) our complete buffer-sharing system is a max–plus linear system. Then with the definition of (2) and some (but tedious) algebra, (4) and (5) can be translated by using the conventional algebra.

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