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Does avoiding bad voting rules result in good ones?

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ABSTRACT

Distance rationalization of voting rules is based on the minimization of the distance to some plausible criterion, such as unanimity or the Condorcet criterion. We propose a new alternative: the optimization of the distance to undesirable voting rules, namely, the dictatorial voting rules. Applying a plausible metric between social choice functions, we obtain two results: (i) the plurality rule minimizes the sum of the distances to the dictatorial rules and can be regarded in some sense as a compromise lying between all dictatorial rules; (ii) the reverse-plurality rule maximizes the distance to the closest dictator.

1. Introduction

The concept of distance rationalization of voting rules has recently been explored by several authors. Given a notion of consensus and a metric (distance function), a voting rule that is rationalizable chooses the alternative that is closest to being a consensus winner. The seminal work was initiated by Farkas and Nitzan [4], who derived the Borda count as the solution of an optimization problem on the set of social choice functions by minimizing the distance from the unanimity principle. Taking other metrics, Nitzan [8] obtained the plurality rule among other rules. The approach of minimizing the distance from a set of profiles with a clear winner such as the unanimous winner, the majority winner, or the Condorcet winner has been developed further by [1,3,6,7], and [10] among others.

All previous works have dealt with the distance rationalizability based on the minimization of the distance to some plausible criterion, such as unanimity or the Condorcet criterion. In contrast, we propose a new alternative, namely, the optimization of the distance to the undesirable dictatorial voting rules, motivated by the classical impossibility results of Arrow [2] and Gibbard–Satterthwaite [5,9], roughly stating that every voting rule satisfying a subset of reasonable properties leads to dictatorship. In particular, we ask the following question: will we obtain a "good" voting rule if we want to get as close as possible to all dictatorial voting rules or if we get away from the closest dictatorial rule? We investigate this question by employing a quite simple and natural distance function between social choice functions.

By getting as close as possible to all dictatorial rules, we are searching for the rules that minimize the sum of the distances to the dictatorial rules, which is identical to the set of rules choosing a top alternative of a voter in as many cases as possible. We call these rules balanced since they represent a kind of compromise between all dictatorial rules. Using this terminology, we find that the plurality rule and the balanced rule are the same. Therefore, we consider this as a positive result since the plurality rule is the most frequently applied one.

By getting away from the closest dictatorial rule, we are searching for the rules that maximize the distance to the closest dictatorial rule. We refer to these rules as the least dictatorial rules since in some sense they are the furthest from dictatorship, which emerges if the collective outcome is determined by a dictatorial rule. In particular, any other rule in the space of voting rules lies closer to at least one of the dictatorial rules than any of the least dictatorial rules. We find that our goal results in a quite unpleasant rule, which we call the reverse-plurality rule, violating properties like unanimity or monotonicity. Therefore, we consider our second main result as a negative one in the sense that we obtain an undesirable rule. However, based on our result, from a philosophical point of view, one could argue that eliminating the 'dictatorial ingredient' from voting rules completely should not be our goal.

Furthermore, we investigate the relationship between minimizing (maximizing) the sum of distances and minimizing (maximizing) the minimum of distances in our objective function.

The plan of the paper is as follows. Section 2 introduces our framework, Section 3 describes our main results, and, finally, Section 4 provides concluding remarks and mentions possible future research directions.

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2. The framework

Let $A = \{1, \dots, m\}$ be the set of alternatives and $N = \{1, \dots, n\}$ be the set of voters. We shall denote by $\mathcal{P}$ the set of all linear orderings (irreflexive, transitive and total binary relations) on A and by $\mathcal{P}^n$ the set of all preference profiles. If $\succ \in \mathcal{P}^n$ and $i \in N$, then $\succ_i$ is the preference ordering of voter i over A.

Definition 1. A mapping $f : \mathcal{P}^n \to A$ that selects the winning alternative is called a *social choice function*, henceforth, SCF.

Note that our definition of an SCF does not allow for possible ties, in which case a fixed tie-breaking rule will be employed. A tie-breaking rule $\tau : \mathcal{P}^n \to \mathcal{P}$ maps preference profiles to linear orderings on A, which will be only employed when a formula does not determine a unique winner. If there are more alternatives chosen by a formula 'almost' specifying an SCF, then the highest ranked alternative is selected, based on the given tie-breaking rule among tied alternatives. In particular, anonymous tie-breaking rules will play a central role in our analysis.

We will also allow for domain restrictions, since for some preference profiles we may prescribe certain outcomes, which are plausible. Let $\mathcal{S} \subseteq \mathcal{P}^n$ be a subdomain on which the outcome is already prescribed by some externally chosen principle. Then the values of a SCF have to be specified only on $\overline{\mathcal{S}}$, where $\overline{\mathcal{S}} = \mathcal{P}^n \setminus \mathcal{S}$, and therefore we only need to consider SCFs restricted to $\overline{\mathcal{S}}$. For instance, for profiles with a Condorcet winner denoted by $\mathcal{S}_c$, we may only consider Condorcet consistent SCFs; or for profiles with a majority supported alternative, denoted by $\mathcal{S}_m$, we may require that the majority winner should be chosen. We consider the following type of domain restriction.

Definition 2. A domain restriction $\mathcal{S} \subseteq \mathcal{P}^n$ is called *anonymous* if for any bijection $\sigma : N \to N$ we have for all $(\succ_1, \dots, \succ_n) \in \mathcal{P}^n$ that $(\succ_1, \dots, \succ_n) \in \mathcal{S}$ implies $(\succ_{\sigma^{-1}(1)}, \dots, \succ_{\sigma^{-1}(n)}) \in \mathcal{S}$.

It can be verified that if $\mathcal{S}$ is anonymous, then also $\overline{\mathcal{S}}$ is anonymous. If $\mathcal{S} = \emptyset$, we have the case of an unrestricted domain. It is easy to see that $\mathcal{S}_c$ and $\mathcal{S}_m$ are anonymous. The introduction of domain restrictions results in a more general framework.

Let $\mathcal{F} = A^{\mathcal{P}^n}$ be the set of SCFs and $\mathcal{F}^{an} \subset \mathcal{F}$ be the set of anonymous voting rules. The subset of $\mathcal{F}$ consisting of the dictatorial rules will be denoted by $\mathcal{D} = \{d_1, \dots, d_n\}$, where d_i is the dictatorial rule with voter i as the dictator. In order to define several optimization problems related to dictatorial rules we will employ the following distance function between SCFs:

$$\rho_{\mathcal{S}}(f, g) = \#\{\succ \in \overline{\mathcal{S}} \mid f(\succ) \neq g(\succ)\}, \tag{2.1}$$

where f, g are SCFs and $\rho_S(f, g)$ stands for the number of profiles on which f and g choose different alternatives within $\overline{\mathcal{S}}$. It can be checked that ρ_S specifies a metric over the set of SCFs restricted to $\overline{\mathcal{S}}$. If $S = \emptyset$, we simply write $\rho(f, g)$. Since in case of SCFs we only care about the chosen outcome (and not about a social ranking), and we do not assume any kind of structure on the set of alternatives A, it appears natural that we count the number of profiles on which f and g differ. We discuss some possible extensions in Section 4.

We specify the set of least dictatorial rules by those ones which are the furthest away from the closest dictatorial rule, which means that we are maximizing the minimum of the distances to the dictators.

Definition 3. We define the set of *least dictatorial rules* for domain restriction $\mathcal{S}$ by

$$\mathcal{F}_{ld}(\mathcal{S}) = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \min_{i \in N} \rho_{\mathcal{S}}(f, d_i) \geq \min_{i \in N} \rho_{\mathcal{S}}(f', d_i) \right\}$$

in general and by

$$\mathcal{F}^{an}_{ld}(\mathcal{S}) = \left\{ f \in \mathcal{F}^{an} \mid \forall f' \in \mathcal{F}^{an} : \min_{i \in N} \rho_{\mathcal{S}}(f, d_i) \geq \min_{i \in N} \rho_{\mathcal{S}}(f', d_i) \right\}$$

over the set of anonymous voting rules.

When defining least dictatorial rules based on the distance function ρ_S, we could have taken the average distance, or equivalently the sum of the distances from the dictators. However, we feel that if we would like to be 'least dictatorial', we should be more concerned about the closest dictatorial rule. Nevertheless, we will consider the other possibility at the end of this section and for anonymous SCFs it will turn out that we will obtain the same rules.

An alternative approach to getting as far away from the closest dictator as possible would be getting as close as possible to all dictators at the same time, which could be considered as a kind of neutral or balanced solution with respect to all dictators and, in this sense, as a kind of desirable solution. For simplicity reasons, we will minimize the sum of the distances to the n dictators.

Definition 4. We define the set of *balanced rules* for domain restriction $\mathcal{S}$ by

$$\mathcal{F}_b(\mathcal{S}) = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \sum_{i \in N} \rho_{\mathcal{S}}(f, d_i) \leq \sum_{i \in N} \rho_{\mathcal{S}}(f', d_i) \right\}$$

in general and by

$$\mathcal{F}^{an}_b(\mathcal{S}) = \left\{ f \in \mathcal{F}^{an} \mid \forall f' \in \mathcal{F}^{an} : \sum_{i \in N} \rho_{\mathcal{S}}(f, d_i) \leq \sum_{i \in N} \rho_{\mathcal{S}}(f', d_i) \right\}$$

over the set of anonymous voting rules.

An equivalent formulation of balanced rules, stating that these rules maximize the number of cases in which a top alternative of a voter is chosen, is derived at the beginning of Section 3.

Instead of looking for the rules which are the furthest away from the closest dictatorial rule we could consider the rules which are the closest ones to the furthest dictatorial rule, which means that we are minimizing the maximum of the distances to the dictators.

Definition 5. We define the set of *minmax* rules for domain restriction $\mathcal{S}$ by

$$\mathcal{F}_{\min\max}(\mathcal{S}) = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \max_{i \in N} \rho_{\mathcal{S}}(f, d_i) \leq \max_{i \in N} \rho_{\mathcal{S}}(f', d_i) \right\}$$

in general and by

$$\mathcal{F}^{an}_{\min\max}(\mathcal{S}) = \left\{ f \in \mathcal{F}^{an} \mid \forall f' \in \mathcal{F}^{an} : \max_{i \in N} \rho_{\mathcal{S}}(f, d_i) \leq \max_{i \in N} \rho_{\mathcal{S}}(f', d_i) \right\}$$

over the set of anonymous voting rules.

In relation to the definition of balanced rules, we obtain the reverse-balanced rules by getting furthest from all dictators at the same time. In particular, we maximize the sum of the distances to the n dictators.

Definition 6. We define the set of *reverse-balanced rules* for domain restriction $\mathcal{S}$ by

$$\mathcal{F}_{rb}(\mathcal{S}) = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \sum_{i \in N} \rho_{\mathcal{S}}(f, d_i) \geq \sum_{i \in N} \rho_{\mathcal{S}}(f', d_i) \right\}$$

in general and by

$$\mathcal{F}^{an}_{rb}(\mathcal{S}) = \left\{ f \in \mathcal{F}^{an} \mid \forall f' \in \mathcal{F}^{an} : \sum_{i \in N} \rho_{\mathcal{S}}(f, d_i) \geq \sum_{i \in N} \rho_{\mathcal{S}}(f', d_i) \right\}$$

over the set of anonymous voting rules.