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On functional limit theorems for the cumulative times in alternating renewal processes

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ABSTRACT

We provide new proofs for two functional central limit theorems, and prove strong approximations for the cumulative "on" times in alternating renewal processes. The proofs rely on a first-passage-time representation of the cumulative "on" time process. As an application, we establish strong approximations for the queueing process in a single-server fluid queue with "on–off" sources.

1. Introduction

Consider an alternating renewal process $N = \{N(t) : t \geq 0\}$ with i.i.d. alternating "on–off" cycles $\{(U_i, V_i) : i \in \mathbb{N}\}$, where the U_i and V_i are "on" and "off" durations in the ith cycle, $i \in \mathbb{N}$. Assume that the process starts at the beginning of an "on" period. Let $m_u = E[U_1] \in (0, \infty)$ and $m_v = E[V_1] \in (0, \infty)$, and $\sigma_u^2 = \mathrm{Var}(U_1) < \infty$ and $\sigma_v^2 = \mathrm{Var}(V_1) < \infty$. Let $T_i = \sum_{k=1}^{i}(U_k + V_k)$ for $i \in \mathbb{N}$ and $T_0 \equiv 0$. Then $N(t) = \max\{i \geq 0 : T_i \leq t\}$ for $t \geq 0$. Define the indicator process $\xi = \{\xi(t) : t \geq 0\}$ by

$$\xi(t) := \begin{cases} 1 & \text{if } T_i \leq t < T_i + U_{i+1}, \\ 0 & \text{if } T_i + U_{i+1} \leq t < T_{i+1}, \end{cases}$$

for each $i \in \mathbb{N}$. When $\xi(t) = 1$, the process is in the "on" period and otherwise the process is in the "off" period. Define the cumulative "on" and "off" processes $X = \{X(t) : t \geq 0\}$ and $Y = \{Y(t) : t \geq 0\}$, respectively, by

$$X(t) := \int_0^t \mathbf{1}(\xi(s) = 1)ds = \int_0^t \xi(s)ds, \tag{1.1}$$

$$Y(t) := \int_0^t \mathbf{1}(\xi(s) = 0)ds = \int_0^t (1 - \xi(s))ds = t - X(t).$$

We focus on the analysis of the cumulative "on" time process X. It is well known (see, e.g., Example 3.6(A) of Section 3.6 in [18]) that $\lim_{t\to\infty} \frac{E[X(t)]}{t} = \gamma_u$ and $\lim_{t\to\infty} \frac{E[Y(t)]}{t} = \gamma_v = 1 - \gamma_u$, where

$$\gamma_u := \frac{m_u}{m_u + m_v}. \tag{1.2}$$

Representation of the cumulative "on" time as the first passage time. The process X can be represented as the first passage time for the random walk associated with the "on–off" cycle times, as observed in [23]. Let $N_u = \{N_u(t) : t \geq 0\}$ be defined by

$$N_u(t) := \max\{k \geq 0 : T_{u,k} \leq t\}, \tag{1.3}$$

with $T_{u,k} := \sum_{i=1}^{k} U_i,\ k \in \mathbb{N},\ T_{u,0} := 0$. Define the compound process $Z_u = \{Z_u(t) : t \geq 0\}$ by

$$Z_u(t) := \sum_{k=1}^{N_u(t)} V_k, \quad t \geq 0. \tag{1.4}$$

Then we can write $X(t)$ directly as $X(t) = \inf\{s > 0 : Z_u(s) > t - s\}$ for $t \geq 0$. Now define an auxiliary process $\check{Z}_u = \{\check{Z}_u(t) : t \geq 0\}$ by

$$\check{Z}_u(t) := Z_u(t) + t, \quad t \geq 0. \tag{1.5}$$

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Thus, we obtain the following representation of the process X as the first passage time of the process $\check{Z}_u$:

$$X(t) = \inf\{s > 0 : \check{Z}_u(s) > t\}, \quad t \geq 0. \tag{1.6}$$

In this paper, we first review two functional central limit theorems (FCLTs) for the cumulative "on" time process X and provide new proofs for these FCLTs (Theorems 2.1 and 2.2 in Section 2). In Theorem 2.1, the "on" and "off" times are of the same order and the result is stated in [21, Theorem 8.3.1] (ours is a slight modification) and its proof is given in Section 5.3 of [22]. That proof applies Theorem 12.5.1 (iv) of [21] by controlling the oscillations of the cumulative "on" time process in the Skorohod M_1 topology. Our new proof takes advantage of the first passage time representation in (1.6) and thus applies the continuous mapping theorem for the inverse mapping with centering [21, Theorem 13.7.2]. This result has been used in establishing FCLTs for the queues with "on–off" sources (see, e.g., [20] and a good review in Section 8 of [21]).

In Theorem 2.2, the "on" and "off" times are of different orders, in particular, the "off" times are asymptotically negligible comparing with the "on" times. The result has been used in queueing systems with service interruptions and server vacations for single-server queues and networks [3,11,12,21]. A similar result is also used for many-server queueing systems with service interruptions [15,14,16,17]. This theorem can be proved with the argument as in the proof of Theorem 14.7.3 in [21]. The proof can also be done with an explicit construction of the parametric representations for the Skorohod M_1 topology (see Section 5.4 in [16]). Here we provide a new proof by applying the continuous mapping theorem to the inverse mapping with centering using the representation in (1.6). The new proofs for these two FCLTs for the cumulative "on" time processes provide important insights on their understanding and future applications.

We prove the strong approximations for the cumulative "on" time processes (Theorem 3.1). Although strong approximations for renewal processes have been well studied and applied in queueing theory [1,4–6,8–10,19], strong approximations for the cumulative "on" time processes in alternating renewal processes have remained open in the literature. The first-passage-time representation of the cumulative "on" time process in (1.6) plays a key role in establishing the strong approximations, since some existing results and proof techniques in [4,5] on renewal processes and the inverse mapping can be applied and/or adapted for our purpose. In Theorem 3.1, we obtain the probability bounds and almost sure properties under the condition of either finite moment generating functions of the "on" and "off" times in a neighborhood of zero or their finite moments of order higher than two.

As an application, the strong approximations of the cumulative "on" time process are applied to a single-server fluid queue with "on–off" sources. Under the proper assumptions on the strong approximations of the input processes, we obtain the strong approximations of the queueing process by a reflected Brownian motion in the critically loaded regime or a Brownian motion in the overloaded regime (Theorem 3.2). Heavy-traffic approximations for fluid queues with "on–off" sources have been well studied in the literature (see a good review in Section 8 of [21]). However, strong approximations for fluid queues with "on–off" sources have remained open. To the best of our knowledge, Theorem 3.2 is the first result on this subject.

1.1. Notation

We use $\mathbb{R}^k$ (and $\mathbb{R}^k_+$), $k \geq 1$, to denote real-valued k-dimensional (nonnegative) vectors, and write $\mathbb{R}$ and $\mathbb{R}_+$ for $k = 1$. Let $\mathbb{N}$ denote the natural numbers. For $x, y \in \mathbb{R}$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Let $\mathbb{D}^k = \mathbb{D}([0,\infty), \mathbb{R}^k)$ denote the $\mathbb{R}^k$-valued function space of all right continuous functions on $[0,\infty)$ with left limits everywhere in $(0,\infty)$. Denote $\mathbb{D} \equiv \mathbb{D}^1$. Let $(\mathbb{D}, J_1)$ and $(\mathbb{D}, M_1)$ denote the space $\mathbb{D}$ equipped with Skorohod J_1 and M_1 topology, respectively. Let $(\mathbb{D}_k, J_1) = (\mathbb{D}, J_1) \times \cdots \times (\mathbb{D}, J_1)$ be the k-fold product of $(\mathbb{D}, J_1)$ with the product topology. Similarly, let $(\mathbb{D}_k, M_1) = (\mathbb{D}, M_1) \times \cdots \times (\mathbb{D}, M_1)$ be the k-fold product of $(\mathbb{D}, M_1)$ with the product topology. Notations $\to$ and $\Rightarrow$ mean convergence of real numbers and convergence in distributions. Let "$\stackrel{d}{=}$" denote "equal in distribution" and ":=" be "definition by equation". The abbreviation *a.s.* means *almost surely*. All random variables and processes are defined on a common probability space $(\Omega, \mathcal{F}, P)$.

2. Functional central limit theorems

In this section we state the two FCLTs for the diffusion-scaled processes of X and provide new proofs for them. We index the quantities and processes with n and use n as a scaling parameter, and let $n \to \infty$.

2.1. "On" and "off" times of the same order

We assume that the "on" and "off" times $\{(U^n_k, V^n_k) : k \in \mathbb{N}\}$ are of the same order, and for the simplicity of exposition, we set them to be independent of n in the scaling below. Note that the random vectors $\{(U^n_k, V^n_k) : k \in \mathbb{N}\}$ are i.i.d. and each pair U^n_k and V^n_k, $k \in \mathbb{N}$, can be correlated.

Define the partial sums associated with the "on" and "off" times $S_{u,n} := \sum_{k=1}^n U_k$ and $S_{v,n} := \sum_{k=1}^n V_k$ for $n \in \mathbb{N}$, and the corresponding processes: for each $n \in \mathbb{N}$, $S_{u,n}(t) := \sum_{k=1}^{\lfloor nt \rfloor} U_k$ and $S_{v,n}(t) := \sum_{k=1}^{\lfloor nt \rfloor} V_k$ for $t \geq 0$. We make the following assumption on these partial sum processes. Let $Disc(\mathcal{X})$ be the random set of discontinuities in $\mathbb{R}_+$ of any stochastic process $\mathcal{X}$.

Assumption 1. There exist positive constants $\alpha_u \in (1, 2]$ and $\alpha_v \in (1, 2]$ and stochastic processes $\tilde{S}_u$ and $\tilde{S}_v$ such that $P(Disc(\tilde{S}_u) \cap Disc(\tilde{S}_v) = \emptyset) = 1$, and $(\tilde{S}_{u,n}, \tilde{S}_{v,n}) \Rightarrow (\tilde{S}_u, \tilde{S}_v)$ in $(\mathbb{D}_2, M_1)$ as $n \to \infty$, where the processes $\tilde{S}_{u,n} = \{\tilde{S}_{u,n}(t) : t \geq 0\}$ and $\tilde{S}_{v,n} = \{\tilde{S}_{v,n}(t) : t \geq 0\}$ are defined by $\tilde{S}_{u,n}(t) := n^{-1/\alpha_u}(S_{u,n}(t) - m_u nt)$ and $\tilde{S}_{v,n}(t) := n^{-1/\alpha_v}(S_{v,n}(t) - m_v nt)$ for $t \geq 0$.

Define the diffusion-scaled processes $\tilde{X}^n = \{\tilde{X}^n(t) : t \geq 0\}$ by $\tilde{X}^n(t) := n^{-1/(\alpha_u \wedge \alpha_v)}(X(nt) - \gamma_u nt)$ for $t \geq 0$. We prove the following FCLT for the processes $\tilde{X}_n$.

Theorem 2.1. *Under Assumption 1, $\tilde{X}^n \Rightarrow \tilde{X}$ in $(\mathbb{D}, M_1)$ as $n \to \infty$, where the limit process $\tilde{X} = \{\tilde{X}(t) : t \geq 0\}$ is given by*

$$\tilde{X}(t) := \begin{cases} -\gamma_u \tilde{S}_v(m_u^{-1}\gamma_u t) & \text{if } \alpha_u > \alpha_v, \\ \gamma_v \tilde{S}_u(m_u^{-1}\gamma_u t) & \text{if } \alpha_u < \alpha_v, \\ -\gamma_u \tilde{S}_v(m_u^{-1}\gamma_u t) + \gamma_v \tilde{S}_u(m_u^{-1}\gamma_u t) & \text{if } \alpha_u = \alpha_v. \end{cases}$$

Proof. Define the diffusion-scaled process $\check{Z}^n_u = \{\check{Z}^n_u(t) : t \geq 0\}$ by $\check{Z}^n_u(t) := n^{-1/(\alpha_u \wedge \alpha_v)}\left(\check{Z}_u(nt) - \gamma_u^{-1}nt\right)$ for $t \geq 0$. It follows from simple algebra that for each $t \geq 0$, $\check{Z}^n_u(t) = n^{-1/(\alpha_u \wedge \alpha_v)}\left(Z_u(nt) - \frac{m_v}{m_u}nt\right) = n^{-1/(\alpha_u \wedge \alpha_v)}\sum_{i=1}^{N_u(nt)}(V_i - m_v) + m_v n^{-1/(\alpha_u \wedge \alpha_v)}(N_u(nt) - m_u^{-1}nt)$. By the continuity of the inverse mapping with centering [21, Theorem 13.7.2] and the convergence of $\tilde{S}_{u,n} \Rightarrow \tilde{S}_u$, we obtain that $n^{-1/\alpha_u}(N_u(nt) - m_u^{-1}nt) \Rightarrow -m_u^{-1}\tilde{S}_u(m_u^{-1}t)$ in $(\mathbb{D}, M_1)$ as $n \to \infty$. This also implies that $n^{-1}N_u(nt) \Rightarrow m_u^{-1}t$ in $(\mathbb{D}, M_1)$ as $n \to \infty$. By the convergence