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Duality, area-considerations, and the Kalai–Smorodinsky solution

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ABSTRACT

We introduce a new solution concept for 2-person bargaining problems, which can be considered as the dual of the Equal-Area solution (EA) (see Anbarcı and Bigelow (1994)). Hence, we call it the Dual Equal-Area solution (DEA). We show that the point selected by the Kalai–Smorodinsky solution (see Kalai and Smorodinsky (1975)) lies in between those that are selected by EA and DEA. We formulate an axiom – *area-based fairness* – and offer three characterizations of the Kalai–Smorodinsky solution in which this axiom plays a central role.

1. Introduction

In a bargaining problem two players need to agree on a utility allocation from a feasible set of allocations, $S \subset \mathbb{R}^2_+$. Failure to reach an agreement leads to a status quo utility of zero for each player. A bargaining solution describes how the players solve every conceivable bargaining problem; formally, a solution is a function that chooses a unique point from every such S. One basic requirement on a bargaining solution is fairness, in the sense that whenever more options become feasible, no one should get hurt. Since a bargaining problem is an infinite object, there are many ways of defining how "more options become feasible". [4] proposed the following fairness requirement, which is based on the area of S: if the area of S increases, no one should get hurt. Moreover, they showed that there is a unique solution (*equal-area solution*) which satisfies this property (*area monotonicity*) and strong Pareto optimality on the domain of convex problems. The equal-area solution assigns to each S the point on its frontier, x, such that the line segment between the origin and x splits S into two parts of equal areas. Informally, the equal-area solution can be considered as an application of the egalitarian principle on an area measure of concessions. [2,5], and [3] provided noncooperative and dynamic foundations for the equal-area solution.

We introduce a new concept into the bargaining literature – *duality* – and apply it to the equal-area solution. The resulting solution is the *dual equal-area solution*. This solution applies the egalitarian principle on an area measure of aspirations. We formulate a requirement – *area-based fairness* – which stipulates that when these two solutions propose the same point, then this point should be chosen as the solution point. We derive three characterizations of the *Kalai–Smorodinsky solution* (see [9]) on the basis of area-based fairness and some standard axioms.

Section 2 describes the bargaining model. Section 3 introduces duality. Section 4 introduces area-based fairness. The characterization results are in Section 5. Finally, Section 6 concludes.

2. The bargaining model: definitions

The following is a simple version of the bargaining model in [10]. A *bargaining problem* is a compact and comprehensive set $S \subset \mathbb{R}^2_+$ that contains $\mathbf{0} \equiv (0, 0)$ as well as some x with $x > \mathbf{0}$, where vector inequalities are defined as follows: uRv if and only if $u_i R v_i$ for each i, for both $R \in \{>, \geq\}$, and $u \gneq v$ if and only if $u \geq v$ and $u \neq v$. The set S (the *feasible set*) is a menu of utility-pairs out of which a single point needs to be agreed upon. Agreement on x means that the bargaining problem is resolved and each agent i obtains the utility x_i; failure to reach an agreement leads to the status quo utilities, $\mathbf{0}$. The assumption $S \cap \mathbb{R}^2_{++} \neq \emptyset$ implies that

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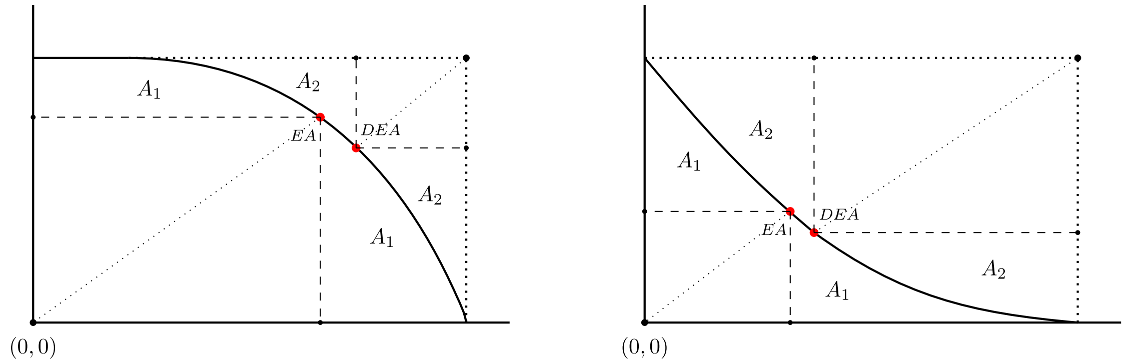

Fig. 1. EA and DEA in convex and non-convex problems.

the status quo can be abandoned in a way that makes both agents better off (relative to the status quo). The assumption that S is comprehensive — namely, that $x \in S \Rightarrow y \in S$ for every y that satisfies $\mathbf{0} \leq y \leq x$ – means that utilities can be freely disposed (down to the status quo levels). Given a problem S, the *ideal point* of S, $a(S)$, is defined by $a_i(S) \equiv \max\{s_i : s \in S\}$. The number $a_i(S)$ is called agent i's *ideal payoff*. The collection of problems is denoted by $\mathcal{B}$. Let $\mathcal{B}^{co} \equiv \{S \in \mathcal{B} : S \text{ is convex}\}$.

A *solution* on a bargaining domain $\mathcal{D} \subset \mathcal{B}$ is a function $f: \mathcal{D} \to \mathbb{R}^2_+$ that satisfies $f(S) \in S$ for every $S \in \mathcal{D}$.

Given a function $\psi: \mathbb{R}^2 \to \mathbb{R}^2$ and a set $C \subset \mathbb{R}^2$, we let $\psi \circ C \equiv \{\psi \circ c : c \in C\}$. A solution f is *scale covariant* if for every problem S and every pair of positive linear transformations, $l = (l_1, l_2)$, it is true that $f(l \circ S) = l \circ f(S)$. A solution f satisfies *contraction independence* if $f(T) \in S \subset T \Rightarrow f(S) = f(T)$. *Contraction independence* has first been introduced by [10] (under the name *independence of irrelevant alternatives*). *Restricted contraction independence*, introduced by [12], narrows the scope of the axiom: it imposes the requirement of *contraction independence* only on pairs of problems, $\{S, T\}$, that in addition to the condition of *contraction independence* share the same ideal point. Another axiom that we will use and that also imposes a restriction on nested problems with a common ideal point is *restricted monotonicity* (introduced by [13]). It requires $[S \subset T \& a(S) = a(T)] \Rightarrow f(S) \leq f(T)$. Finally, f is *continuous* if for each sequence of problems $\{S_k\}$ that converges to a problem S in the Hausdorff topology, $f(S_k)$ converges to $f(S)$.

Given $v > \mathbf{0}$, the *comprehensive hull* of $\{v\}$ is $\text{comp}\{v\} \equiv \{x \in \mathbb{R}^2_+ : x \leq v\}$. Given a set $A \subset \mathbb{R}^2_+$, $\bar{A}$ denotes the closure of A.

3. Duality

Given a problem S with $S \neq \text{comp}\{a(S)\}$, we define the *dual problem of* S as $D(S) \equiv \overline{\text{comp}\{a(S)\} \setminus S}$.

At an informal level, one can think of the following story as underlining a bargaining problem: the agents "start" at the origin, and they need to move forward, towards the frontier, and reach an agreed-upon point. Then, the informal story corresponding to the dual problem is that the agents "start" at the ideal point and they need to concede to move towards the frontier and reach a feasible point.

We define the duality transformation, $\phi_S = (\phi_{S,1}, \phi_{S,2})$, by $\phi_{S,i}(x) \equiv -(x - a_i(S))$. The transformations $\phi_{S,1}$ and $\phi_{S,2}$ shift the ideal point to the origin and then "mirror" the resulting set, carrying it from the south-west quadrant to the north-east one. Note that for every $S \in \mathcal{B}$ such that $S \neq \text{comp}\{a(S)\}$ we have $\phi_S \circ D(S) \in \mathcal{B}$. More generally, a bargaining domain $\mathcal{D}$ is *closed under duality* if $[S \in \mathcal{D}] \& [S \neq \text{comp}\{a(S)\}] \Rightarrow \phi_S \circ D(S) \in \mathcal{D}$. In this paper, we consider two domains: the grand domain $\mathcal{B}$ and its subset $\mathcal{B}^{co}$. The former is closed under duality, whereas the latter is not.

Let $\mathcal{D}$ be a domain such that (i) it is closed under duality and (ii) no $S \in \mathcal{D}$ contains a boundary with a segment parallel to an axis. Let f and $\tilde{f}$ be two solutions on $\mathcal{D}$. The solution $\tilde{f}$ is the *dual of* f *on* $\mathcal{D}$ if the following holds for all $S \in \mathcal{D}$:

$$\tilde{f}(S) = \phi_S^{-1} \circ f(\phi_S \circ D(S)), \tag{1}$$

where $\phi_{S,i}^{-1}$ is the inverse of $\phi_{S,i}$. In words, duality says that applying $\tilde{f}$ to S is same as "standing at $a(S)$" and applying f to $D(S)$. Requirement (i) makes sure that $\phi_S \circ D(S)$ on the RHS of (1) belongs to $\mathcal{D}$. Requirement (ii) guarantees that $\mathbf{0}$ is the "ideal point" of $D(S)$.

The *egalitarian* solution (see [8]) and the *equal-loss* solution (see [6]) are two solutions that are duals of each other. The former assigns to each S the intersection point of its (weak) Pareto frontier and the 45°-line, whereas the latter assigns to each S the intersection point of its (weak) Pareto frontier and the 45°-line drawn from $a(S)$. On the other hand, the *Kalai–Smorodinsky solution*, which is defined by $KS(S) \equiv \max\{\lambda : \lambda a(S) \in S\} \times a(S)$ is dual to itself (i.e., Eq. (1) holds with $f = \tilde{f} = KS$) on the domain of problems with a boundary that does not contain a segment parallel to an axis.

4. Area-based fairness

The *equal-area solution* (see [4]) assigns to each S the point of its Pareto frontier, x, such that the segment $\text{conv}\{\mathbf{0}, x\}$ splits S into two subsets of equal areas (see Fig. 1). In this paper, we introduce the dual of this solution, which we call the *dual equal-area solution*. For each S, this solution assigns the Pareto efficient x such that $\text{conv}\{x, a(S)\}$ splits $D(S)$ into two subsets of equal areas (see Fig. 1). We denote the equal-area solution and its dual by *EA* and *DEA*, respectively.

As we mentioned in the Introduction, for convex S, *EA* is the only strongly Pareto optimal solution satisfying the requirement that no one loses when the area of S increases. Analogously, *DEA* satisfies the following property: when the area of S decreases and the ideal point is unchanged, no one benefits. *DEA* satisfies some standard axioms such as Pareto optimality, symmetry, and scale covariance.

EA and *DEA* demonstrate that there are at least two ways to define egalitarianism on the basis of areas: the original way of [4] and the dual way of *DEA* introduced here. We seek to merge them into a single fairness criterion. We believe that any reasonable merging would adhere to the following requirement:

Definition 1. A solution f satisfies *area-based fairness* if for each S the following holds:

$$EA(S) = DEA(S) = x \Rightarrow f(S) = x.$$

That is, in those cases where the two solutions agree, the *area-based fairness* property requires this agreed-upon point to be the solution-point.