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Saddle point optimality criteria for mathematical programming problems with equilibrium constraints

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ABSTRACT

In this paper, we consider a mathematical programming problem with equilibrium constraints (MPEC). We formulate the Lagrange type dual model for the MPEC and establish weak and strong duality results under convexity assumptions. Further, we investigate the saddle point optimality criteria for the MPEC. We also illustrate our results by an example.

1. Introduction

In this paper, we consider the following mathematical programming problem with equilibrium constraints:

$$\begin{array}{ll} \text{MPEC} & \min f(x) \\ \text{s.t.} & g(x) \leq 0, \quad h(x) = 0, \\ & G(x) \geq 0, \quad H(x) \geq 0, \quad G(x)^T H(x) = 0, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable function, $g : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^n \to \mathbb{R}^q$, $G : \mathbb{R}^n \to \mathbb{R}^l$, $H : \mathbb{R}^n \to \mathbb{R}^l$ are functions with continuously differentiable components and ^{T} indicates the transpose.

Ye [16] has given a simple proof of the Mordukhovich (M-) stationary condition and showed that it is also sufficient condition for the MPEC under pseudoconvexity assumption on objective function and quasiconvexity on constraints. Further, Ye [16] obtained some new constraint qualifications for the MPEC as generalization of existing constraints qualifications for the MPEC. For further studies on stationary point conditions and constraint qualifications for MPEC, we refer to [7,5,10,1,2,8] and the references therein.

Recently, Pandey and Mishra [14] and Guu et al. [9] studied Wolfe and Mond–Weir type dual models for MPEC and established weak and strong duality results under generalized convexity assumptions.

Motivated by the above mentioned works, we propose the Lagrange type dual model for the MPEC. We establish weak and strong duality results and give relationships between a primal MPEC solution and a saddle point of the MPEC Lagrangian.

2. Preliminaries

Throughout the paper, we denote by S the feasible region of the MPEC. Given a feasible point $\bar{x} \in S$, we consider the following index sets:

$$\begin{aligned} I_g &:= I(\bar{x}) = \{i \in \{1, 2, \dots, p\} : g_i(\bar{x}) = 0\}, \\ \alpha &:= \alpha(\bar{x}) = \{i \in \{1, 2, \dots, l\} : G_i(\bar{x}) = 0, H_i(\bar{x}) > 0\}, \\ \beta &:= \beta(\bar{x}) = \{i \in \{1, 2, \dots, l\} : G_i(\bar{x}) = 0, H_i(\bar{x}) = 0\}, \\ \gamma &:= \gamma(\bar{x}) = \{i \in \{1, 2, \dots, l\} : G_i(\bar{x}) > 0, H_i(\bar{x}) = 0\}. \end{aligned}$$

The following concept of tangent cone is well known in optimization (see, [15]).

Definition 2.1. The tangent cone of a set Ω at $\bar{x} \in cl\Omega$ is the closed cone defined by

$$T(\bar{x}) := \{d \in \mathbb{R}^n : \exists t_n \downarrow 0, d_n \to d \text{ s.t. } \bar{x} + t_n d_n \in \Omega\}.$$

Since the standard Abadie constraint qualification is not satisfied for the MPEC, the following modified Abadie constraint qualification for the MPEC was introduced by Flegel and Kanzow [4].

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Definition 2.2 (*MPEC Abadie CQ*)**.** Let $\bar{x}$ be a feasible point of the MPEC. We say that MPEC Abadie constraint qualification holds at $\bar{x}$ if

$$T^{lin}_{MPEC}(\bar{x}) = T(\bar{x}),$$

where

$$\begin{aligned} T^{lin}_{MPEC}(\bar{x}) := \big\{ d \in \mathbb{R}^n : \nabla g_i(\bar{x})^T d \le 0,\ \forall i \in I_g,\\ \nabla h_i(\bar{x})^T d = 0,\ \forall i = 1, 2, \ldots, q,\\ \nabla G_i(\bar{x})^T d = 0,\ \forall \alpha,\\ \nabla H_i(\bar{x})^T d = 0,\ \forall i \in \gamma,\\ \nabla G_i(\bar{x})^T d \ge 0,\ \forall i \in \beta,\\ \nabla H_i(\bar{x})^T d \ge 0,\ \forall i \in \beta,\\ \big(\nabla G_i(\bar{x})^T d\big) . \big(\nabla H_i(\bar{x})^T d\big) = 0,\ \forall i \in \beta \big\}, \end{aligned}$$

is the MPEC linearized tangent cone for the MPEC.

The following concept of M-stationary for the MPEC was introduced by Outrata [13], however the concept of M-stationary for optimization problems with variational inequality constraints was first introduced by Ye and Ye [17].

Definition 2.3 (*M-Stationary Point*)**.** A feasible point $\bar{x}$ of the MPEC is called Mordukhovich stationary if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2l}$ such that the following conditions hold:

$$\nabla f(\bar{x}) + \sum_{i=1}^{p} \lambda_i^g \nabla g_i(\bar{x}) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{i=1}^{l} \left[\lambda_i^G \nabla G_i(\bar{x}) + \lambda_i^H \nabla H_i(\bar{x})\right] = 0, \quad (2.1)$$

$$\lambda^g \ge 0, \qquad g(\bar{x})^T \lambda^g = 0, \quad (2.2)$$

$$\lambda_\gamma^G = 0, \qquad \lambda_\alpha^H = 0, \quad (2.3)$$

$$\forall i \in \beta \quad \text{either } \lambda_i^G > 0, \quad \lambda_i^H > 0 \quad \text{or} \quad \lambda_i^G \lambda_i^H = 0. \quad (2.4)$$

The following necessary optimality conditions were derived by Ye [16] under the MPEC Abadie CQ.

Theorem 2.1. *Let $\bar{x}$ be a local optimal solution of the MPEC. Suppose that the MPEC Abadie CQ is satisfied at $\bar{x}$. Then, $\bar{x}$ is M-stationary.*

We know that a differentiable function f defined on a nonempty open convex set $\Omega \subseteq \mathbb{R}^n$ is convex on Ω if and only if for all $x, y \in \Omega$, we have

$$f(y) - f(x) \ge \nabla f(x)^T (y - x).$$

3. Lagrange type duality for MPEC

In this section, we formulate the Lagrange type duality and investigate weak and strong duality results between the corresponding problems.

Let

$$\varphi(\lambda) = \min_{x \in \mathbb{R}^n} L_{MPEC}(x, \lambda),$$

where

$$L_{MPEC}(x, \lambda) := f(x) + \sum_{i=1}^{p} \lambda_i^g g_i(x) + \sum_{i=1}^{q} \lambda_i^h h_i(x) - \sum_{i=1}^{l} \left[\lambda_i^G G_i(x) + \lambda_i^H H_i(x)\right],$$

is the MPEC Lagrangian and $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H)$.

We present the following Lagrange type dual problem depending on a feasible point $x \in S$ for the MPEC:

$$\begin{aligned} &MPECLD(x) && \max \varphi(\lambda^g, \lambda^h, \lambda^G, \lambda^H)\\ &\text{s.t.} && \lambda_i^g \ge 0, \quad i \notin I_g,\\ &&& \lambda_\gamma^G = 0, \quad \lambda_\alpha^H = 0,\\ &&& \text{either} \quad \lambda_i^G > 0,\ \lambda_i^H > 0 \quad \text{or}\\ &&& \lambda_i^G \lambda_i^H = 0, \quad \forall i \in \beta. \end{aligned}$$

Let $S_D(x)$ be the feasible region of the *MPECLD*(x) corresponding to a point $x \in S$. The dual problem *MPECLD*(x) depends on a feasible point $x \in S$ of the MPEC. Now, we present the Lagrange type dual problem which is independent of the MPEC:

$$\begin{aligned} &MPECLD && \max \varphi(\lambda^g, \lambda^h, \lambda^G, \lambda^H)\\ &\text{s.t.} && (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in S_D, \end{aligned}$$

where $S_D = \cap_{x \in S} S_D(x) \neq \phi$ is feasible region of the MPECLD.

Remark 3.1. The feasible region of the MPECLD is the intersection of the feasible regions of the *MPECLD*(x) for all $x \in S$. Hence, a feasible solution of the MPECLD is also a feasible solution of the *MPECLD*(x) for all $x \in S$. Also, MPECLD will serve as a better duality problem than *MPECLD*(x) since it independent of the primal problem.

Theorem 3.1 (*Weak Duality*)**.** *If x and λ are feasible points for the MPEC and the MPECLD(x), respectively, then*

$$f(x) \ge \varphi(\lambda).$$

Proof. Since $x \in S$ and $\lambda \in S_D(x)$ are feasible points for the MPEC and the *MPECLD*(x), respectively, then one has

$$\begin{aligned} \varphi(\lambda) &= \min_{x \in \mathbb{R}^n} L_{MPEC}(x, \lambda)\\ &\le f(x) + \sum_{i=1}^{p} \lambda_i^g g_i(x) + \sum_{i=1}^{q} \lambda_i^h h_i(x) - \sum_{i=1}^{l} \left[\lambda_i^G G_i(x) + \lambda_i^H H_i(x)\right]\\ &\le f(x). \end{aligned}$$

This completes the proof.

Corollary 3.1. *If x and λ are feasible points for the MPEC and the MPECLD, respectively, then*

$$f(x) \ge \varphi(\lambda).$$

Corollary 3.2. *If $\bar{x}$ and $\bar{\lambda}$ are the optimal solutions for the MPEC and the MPECLD, respectively, then*

$$f(\bar{x}) \ge \varphi(\bar{\lambda}).$$

Corollary 3.3. *If $\bar{x}$ and $\bar{\lambda}$ are feasible points for the MPEC and the MPECLD, respectively, and $f(\bar{x}) = \varphi(\bar{\lambda})$, then $\bar{x}$ and $\bar{\lambda}$ are optimal solutions for the MPEC and the MPECLD, respectively.*

The following theorem presents strong duality relation between the MPEC and the *MPECLD*($\bar{x}$) at a local optimal solution $\bar{x}$ of the MPEC.

Theorem 3.2 (*Strong Duality*)**.** *If $\bar{x}$ is a local optimal solution of the MPEC and the MPEC Abadie CQ holds at $\bar{x}$ and $L_{MPEC}(x, \lambda)$ is convex at $\bar{x}$ for all $\lambda \in S_D(\bar{x})$ and for all $x \in \mathbb{R}^n$ then there exists $\bar{\lambda} := (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^{p+q+2l}$ such that $\bar{\lambda}$ is a global optimal solution of the MPECLD($\bar{x}$) and $f(\bar{x}) = \varphi(\bar{\lambda})$.*