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On Khintchine type inequalities for k-wise independent Rademacher random variables

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ABSTRACT

We consider Khintchine type inequalities on the pth moments of vectors of N k-wise independent Rademacher random variables. We show that an analogue of Khintchine's inequality holds, with a constant $N^{1/2-k/2p}$, when k is even. We then show that this result is sharp for $k = 2$; in particular, a version of Khintchine's inequality for sequences of pairwise Rademacher variables *cannot* hold with a constant independent of N. We also characterize the cases of equality and show that, although the vector achieving equality is not unique, it is unique (up to law) among the smaller class of exchangeable vectors of pairwise independent Rademacher random variables. As a fortunate consequence of our work, we obtain similar results for 3-wise independent vectors.

1. Introduction

This short note concerns Khintchine's inequality, a classical theorem in probability, with many important applications in both probability and analysis (see Garling, 2007; Kahane, 1985; Lindenstrauss and Tzafriri, 1996; Milman and Schechtman, 1986; Peskir and Shiryaev, 1995 among others). It states that the L_p norm of the weighted sum of independent Rademacher random variables is controlled by its L_2 norm; a precise statement follows. We say that ε_0 is a Rademacher random variable if $\mathbb{P}(\varepsilon_0 = 1) = \mathbb{P}(\varepsilon_0 = -1) = \frac{1}{2}$. Let $\bar{\varepsilon}_i$, $1 \le i \le N$, be independent copies of ε_0 and $a \in \mathbb{R}^N$. Khintchine's inequality (see, for example, Theorem 2.b.3 in Lindenstrauss and Tzafriri (1996), Theorem 12.3.1 in Garling (2007) or the original work of Khintchine (1923)) states that, for any $p > 0$

$$B(p)\left(\mathbb{E}\left|\sum_{i=1}^N a_i\bar{\varepsilon}_i\right|^2\right)^{\frac{1}{2}} = B(p)\|a\|_2 \le \left(\mathbb{E}\left|\sum_{i=1}^N a_i\bar{\varepsilon}_i\right|^p\right)^{\frac{1}{p}} \le C(p)\,\|a\|_2 = C(p)\left(\mathbb{E}\left|\sum_{i=1}^N a_i\bar{\varepsilon}_i\right|^2\right)^{\frac{1}{2}}. \tag{1}$$

We will mostly be interested in the upper Khintchine inequality; that is, the second inequality in (1). Note here that the upper constant $C(p)$ depends only on p; in particular, it does not depend on N. In what follows, we take $C(p)$ to be the best possible constant in (1). This value is in fact known explicitly (Haagerup, 1981):

$$C(p) = \begin{cases} 1 & 0 < p \le 2 \\ \dfrac{\sqrt{2}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}^{1/p}} & p > 2. \end{cases}$$

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It is natural to ask whether the independence condition can be relaxed; indeed, random vectors with dependent coordinates arise in many problems in probability and analysis (see e.g. Guedon et al. (in press) and the references therein). In this short paper, we are interested in what can be said when the independence assumption on the coordinates is relaxed to pairwise (or, more generally, k-wise) independence.

Definition 1.1. We call an N-tuple $\varepsilon = \{\varepsilon_i\}_{i=1}^N$ of Rademacher random variables a *Rademacher vector*, or (finite) *Rademacher sequence.* For a fixed non-negative integer k, a Rademacher vector is called *k-wise independent* if any subset $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_k}\}$ of size k is mutually independent.

When $k = 2$ in the preceding definition, we will often use the terminology *pairwise independent* in place of 2-wise independent. For more on k-wise independent sequences and their construction, see, for example Derriennic and Klopotowski (1991) and Robertson (1985, 1988).

As it will be useful in what follows, we note that instead of random variables, it is equivalent to consider probability measures P on the set $\{-1, 1\}^N$, where $P = law(\varepsilon)$. The condition that ε is a Rademacher vector is then equivalent to the condition that the projections $law(\varepsilon_i)$ of P onto each copy of $\{-1, 1\}$ are all equal to $P_1 := \frac{1}{2}[\delta_{-1} + \delta_1]$. The k-wise independence condition is equivalent to the condition that the projections $law(\varepsilon_{i_1}, \dots, \varepsilon_{i_k})$ of P onto each k-fold product $\{-1, 1\}^k$ is product measure $\otimes^k P_1$.

An interesting general line of research in probability aims to understand which of the many known properties of mutually independent sequences carry over to the k-wise independent setting; how much independence is actually needed to assert various properties? Some results, including the second Borel–Cantelli lemma and the strong law of large numbers (see, for instance, Etemadi, 1981; Andrews, 1988) hold true for pairwise independent sequences, whereas others, such as the central limit theorem, do not. We found it surprising that little seems to be known about Khintchine's inequality for k-wise independent sequences (except when $k \geq p$, as we discuss briefly below).

It is therefore natural to ask whether Khintchine's inequality holds for k-wise independent Rademacher random variables, and, if not, to understand how badly it fails. More precisely, we focus on the upper Khintchine inequality and define

$$C(N, p, k) = \sup_{\substack{a \in \mathbb{R}^N : \|a\|_2 = 1 \\ \varepsilon \text{ is a } k\text{-wise independent Rademacher vector}}} \left(\mathbb{E} \left| \sum_{i=1}^N a_i \varepsilon_i \right|^p \right)^{1/p}. \tag{2}$$

The questions we are interested in can then be formulated as follows:

1. Is $C(N, p, k)$ bounded as $N \to \infty$, for a fixed $p > k$?
2. If not, what is the growth rate of $C(N, p, k)$?

Note that $C(N, p, k)$ forms a monotone decreasing sequence in k, as the k-dependence constraint becomes increasingly stringent as k grows. Note that, as mutual independence implies k-wise independence for any k, we have $C(N, p, k) \geq C(p)$, where $C(p)$ is the best constant in the classical Khintchine inequality (1).

Some properties of $C(N, p, k)$ are easily discerned. For example, it is straightforward to see that $C(N, 2, k) = 1$. Let us also mention that, when p is an even integer, and $k \geq p$, it is actually a straightforward calculation to show that $C(N, p, k) = C(p)$ is independent of N (that is, Khintchine's inequality for k-wise independent random variables holds with the same constant as in the independence case).

For $k < p$ and even, we first show that $C(N, p, k) \leq C(k)^{k/p} N^{1/2-k/2p}$, by combining a standard interpolation argument with the classical Khintchine inequality and the observation above. This provides some information on the second question above for general k.

We then focus on the $k = 2$ case. We prove that for $p \geq 2$ and N even, $C(N, p, 2) = N^{1/2-1/p}$, providing a negative answer to the first question above. We construct an explicit pairwise independent Rademacher sequence satisfying the equality. Finally, we characterize the cases of equality, and prove that although this equality may be achieved by multiple Rademacher vectors, the one we construct is the unique exchangeable equality case (up to law).

As a fortunate consequence of our work here, we obtain analogous results for $k = 3$. Understanding the $k \geq 4$ case remains an interesting open question.

2. A general estimate on $C(N, p, k)$

We begin by establishing an upper bound on $C(N, p, k)$ via a straightforward interpolation argument.

Proposition 2.1. *For all $p \geq k \geq 2$ and k even, we have $C(N, p, k) \leq C(k)^{k/p} N^{1/2-k/2p}$.*

Proof. Let $\epsilon = (\epsilon_1, \dots, \epsilon_N)$ be a k-wise independent Rademacher vector of length N, and $a = (a_1, \dots, a_N) \in \mathbb{R}^N$. Set $f = |\sum_{i=1}^N a_i \epsilon_i|$, so that f is a function on the underlying probability space. Writing $f^p = f^k f^{p-k}$, we apply Holder's inequality to get

$$\|f\|_p \leq \|f\|_k^{k/p} \|f^{p-k}\|_\infty^{1/p} \tag{3}$$