Statistics and Probability Letters 132 (2018) 46–54

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

A random regularized approximate solution of the inverse problem for Burgers' equation

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ARTICLE INFO

Article history:
Received 26 February 2017
Received in revised form 24 June 2017
Accepted 23 August 2017
Available online 23 September 2017

Keywords:
Inverse problem
Burgers equation
Random approximation

ABSTRACT

In this paper, we find a regularized approximate solution for an inverse problem for Burgers' equation. The solution of the inverse problem for Burgers' equation is ill-posed, i.e., the solution does not depend continuously on the data. The approximate solution is the solution of a regularized equation with randomly perturbed coefficients and randomly perturbed final value and source functions. To find the regularized solution, we use the modified quasi-reversibility method associated with the truncated expansion method with nonparametric regression. We also investigate the convergence rate.

1. Introduction

In this work, we consider a backward in time problem for **1-D Burgers' equation**

$$\begin{cases} \mathbf{u}_t - (A(x,t)\mathbf{u}_x)_x &= \mathbf{u}\mathbf{u}_x + G(x,t), & (x,t) \in \Omega \times (0,T), \\ \mathbf{u}(x,t) &= 0, & x \in \partial\Omega, \\ \mathbf{u}(x,T) &= H(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = (0,\pi)$. Here the coefficient $A(x,t)$ is a C^1 smooth function and $A(x,t) \geq \overline{a} > 0$. The Burgers equation is a fundamental partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow (Kukavica, 2007).

One can see that the term $(A(x,t)\mathbf{u}_x)_x$ is $\Delta\mathbf{u} = \mathbf{u}_{xx}$ if $A = 1$. However, one cannot use spectral methods to study the operator $(A(x,t)\mathbf{u}_x)_x$. So, the problem (1.1) is a challenging one. The second observation is that for the equation $\mathbf{u}_t - (A(x,t)\mathbf{u}_x)_x = f(\mathbf{u},\mathbf{u}_x)$ when $A(x,t)$ is deterministic and $f(\mathbf{u},\mathbf{u}_x) = f(\mathbf{u})$, the problem is a consequence of Theorem 4.1 in our recent paper (Kirane et al., 2017a). However, if $A(x,t)$ is randomly perturbed and $f(\mathbf{u},\mathbf{u}_x)$ depends on $\mathbf{u}$ and $\mathbf{u}_x$ then the problem is again more challenging.

Until now, the deterministic Burgers equation with the randomly perturbed case has not been studied. Hence, the paper is the first study of Burgers' equation backward in time. The inclusion of the gradient term in $\mathbf{u}\mathbf{u}_x$ in the right hand side of Burgers' equation makes Burgers' equation more difficult to study. We need to find an approximate function for $\mathbf{u}\mathbf{u}_x$. This task is nontrivial.

This paper is a continuation of our study of backward problems in the two recent papers (Kirane et al., 2017a, b). In those papers the equations did not have random coefficients in the main equations. The paper (Kirane et al., 2017a) does not consider the random operator. The paper (Kirane et al., 2017b) considers the simple coefficient $A(x,t) = A(t)$ and the source

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https://doi.org/10.1016/j.spl.2017.08.014
0167-7152/

function is $\mathbf{u}-\mathbf{u}^3$. Hence, one can see that Burgers' equation considered here is more difficult since the gradient term in the right hand side and the coefficient $A(x,t)$ depend on both x and t.

It is known that the backward problem mentioned above is ill-posed in general (Kukavica, 2007), i.e., a solution does not always exist. When the solution exists, the solution does not depend continuously on the given initial data. In fact, from a small perturbation of a physical measurement, the corresponding solution may have a large error. This makes the numerical computation troublesome. Hence a regularization is needed. It is well-known that there are some difficulties to study the nonlocal Burger equation. First, by the given form of coefficient $A(x,t)$ in the main equation (1.1), the solution of problem (1.1) cannot be transformed into a nonlinear integral equation. Hence, classical spectral methods cannot be applied. The second reason that makes Burger's equation more difficult to study is the presence of gradient term $\mathbf{u}_x$ in the right hand side. Until now, although there are limited number of results on the backward problem for Burgers' equation (Carasso, 1977; Hao et al., 2015), there are no results for regularizing the problem with random case.

As is well-known, measurements always are given at a discrete set of points and contain errors. These errors may be generated from controllable sources or uncontrollable sources. In the first case, the error is often deterministic. If the errors are generated from uncontrollable sources as wind, rain, humidity, etc., then the model is random. Methods used for the deterministic cases cannot be applied directly to the random case.

In this paper, we consider the following model:

$$\widetilde{H}(x_k)=H(x_k)+\sigma_k\epsilon_k,\quad \widetilde{G}_k(t)=G(x_k,t)+\vartheta\xi_k(t),\quad \text{for}\quad k=\overline{1,n}, \tag{1.2}$$

and

$$\widetilde{A}_k(t)=A(x_k,t)+\overline{\vartheta}\xi_k(t),\quad \text{for}\quad k=\overline{1,n}. \tag{1.3}$$

where $x_k=\pi\frac{2k-1}{2n}$ and ϵ_k are unknown independent random errors. Moreover, $\epsilon_k\sim\mathcal{N}(0,1)$, and $\sigma_k,\ \vartheta,\overline{\vartheta}$ are unknown positive constants which are bounded by a positive constant V_{max}, i.e., $0\le\sigma_k<V_{\max}$ for all $k=1,\ldots,n$. $\xi_k(t)$ are Brownian motions. The noises ϵ_k, $\xi_k(t)$ are mutually independent. Our task is reconstructing the initial data $\mathbf{u}(x,0)$.

We next want to mention about the organization of the paper and our methods in this paper. We prove some preliminary results in Section 2. We state and prove our main result in Section 3. The existence and uniqueness of solution of Eq. (1.1) is an open problem, and we do not investigate this problem here. For the inverse problem, we assume that the solution of Burgers' equation (1.1) exists. In this case its solution is not stable. In this paper we establish an approximation of the backward in time problem for 1-D Burgers' equation (1.1) with the solution of a regularized equation with randomly perturbed equation (2.13). The random perturbation in Eq. (2.13) is explained in Eqs. (1.2), (1.3), (2.13) and (2.14).

2. Some notation

We first introduce notation, and then state our main results in this paper. We define the Dirichlet–Laplacian.

$$Af(x):=-\Delta f(x)=-\frac{\partial^2 f(x)}{\partial x^2}. \tag{2.4}$$

Since A is a linear densely defined self-adjoint and positive definite elliptic operator on the connected bounded domain $\Omega=(0,\pi)$ with Dirichlet boundary condition, the eigenvalues of A satisfy

$$\lambda_0=0<\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_p\le\cdots$$

with $\lambda_p=p^2\to\infty$ as $p\to\infty$. The corresponding eigenfunctions are denoted, respectively, by $\varphi_p(x)=\sqrt{\frac{2}{\pi}}\sin(px)$. Thus the eigenpairs (λ_p,ϕ_p), $p=0,1,2,\ldots$, satisfy

$$\begin{cases} A\varphi_p(x)=-\lambda_p\phi_p(x), & x\in\Omega\\ \phi_p(x)=0, & x\in\partial\Omega. \end{cases}$$

The functions φ_p are normalized so that $\{\phi_p\}_{p=0}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$.
Defining

$$H^{\gamma}(\Omega)=\left\{v\in L^2(\Omega):\sum_{p=0}^{\infty}\lambda_p^{2\gamma}\left|\langle v,\phi_p\rangle\right|^2<+\infty\right\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\Omega)$, then $H^{\gamma}(\Omega)$ is a Hilbert space equipped with norm

$$\|v\|_{H^{\gamma}(\Omega)}=\left(\sum_{p=1}^{\infty}\lambda_p^{2\gamma}\left|\langle v,\phi_p\rangle\right|^2\right)^{1/2}.$$

Now, we describe the main idea of our method. First, we approximate H and G by the approximating functions $\widehat{H}_{\beta_n}$ and $\widehat{G}_{\beta_n}$ that are defined in Theorem 2.1. Next, our task is of finding the approximating operator for $\nabla(A(x,t)\nabla)$. We will not