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Stochastic Processes and their Applications 126 (2016) 3331–3352
stochastic processes and their applications
www.elsevier.com/locate/spa

Fluctuations of linear statistics of half-heavy-tailed random matrices

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Received 21 October 2014; received in revised form 12 April 2016; accepted 27 April 2016
Available online 13 May 2016

Abstract

In this paper, we consider a Wigner matrix A with entries whose cumulative distribution decays as $x^{-\alpha}$ with $2 < \alpha < 4$ for large x. We are interested in the fluctuations of the linear statistics $N^{-1}\operatorname{Tr}\varphi(A)$, for some nice test functions φ. The behavior of such fluctuations has been understood for both heavy-tailed matrices (i.e. $\alpha < 2$) in Benaych-Georges (2014) and light-tailed matrices (i.e. $\alpha > 4$) in Bai and Silverstein (2009). This paper fills in the gap of understanding it for $2 < \alpha < 4$. We find that while linear spectral statistics for heavy-tailed matrices have fluctuations of order $N^{-1/2}$ and those for light-tailed matrices have fluctuations of order N^{-1}, the linear spectral statistics for half-heavy-tailed matrices exhibit an intermediate α-dependent order of $N^{-\alpha/4}$.

MSC: 15A52; 60F05

Keywords: Random matrices; Heavy tailed random variables; Central limit theorem

1. Introduction

Let $A = [a_{ij}]$ be an $N \times N$ Hermitian random matrix whose entries are i.i.d. and let $\lambda_1, \dots, \lambda_N$ be its eigenvalues. It is well known that if the entries of A are duly renormalized,

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http://dx.doi.org/10.1016/j.spa.2016.04.030
0304-4149/

Table 1
Orders of the fluctuations of the r.v. of (1) around its expectation as a function of the exponent α such that $\mathbb{P}(|a_{ij}| > x) \approx x^{-\alpha}$ for x large.

	$\alpha < 2$	$2 < \alpha < 4$	$\alpha > 4$
Order of the fluctuations of (1)	$N^{-1/2}$	$N^{-\alpha/4}$	N^{-1}

then for any continuous bounded test function φ, the random variable

$$\frac{1}{N}\operatorname{Tr}\varphi(A) = \frac{1}{N}\sum_{i=1}^{N}\varphi(\lambda_i) \tag{1}$$

has a deterministic limit, which is equal to the integral of f with respect to the limit spectral distribution of A, namely the semicircle law when the entries have at least a second moment [2,1] and different distributions depending on α if the entries are heavy-tailed with exponent $\alpha \in (0, 2)$ (see [5,14,9]). The rate of convergence of the random variables of (1) to its limit is not usually $\frac{1}{\sqrt{N}}$, as i.i.d. λ_i's would give. In particular, if the entries of A have a fourth moment, then the fluctuations of $\frac{1}{N}\operatorname{Tr}\varphi(A)$ around its expectation have order $\frac{1}{N}$ (see [2,19,24,18,4,3,21,23]). On the other hand, if the entries are heavy-tailed with exponent $\alpha \in (0, 2)$ or Bernoulli with parameter of order N^{-1}, then the fluctuations of $\frac{1}{N}\operatorname{Tr}\varphi(A)$ around its expectation have order $N^{-1/2}$ [6]. This difference of order in the fluctuations is due to the fact that when the entries of A have enough moments, the eigenvalues of A fluctuate very little, as studied by Erdös, Schlein, Yau, Tao, Vu and their co-authors, who analyzed their rigidity in e.g. [15,16,25]. On the other hand, the heavier the tails the more similar to a sparse matrix the (renormalized) matrix A is, and the more independently its eigenvalues behave.

A finite fourth moment means that for large x, $\mathbb{P}(|a_{ij}| > x) \approx x^{-\alpha}$ with $\alpha > 4$, whereas heavy-tailed entries with exponent $\alpha \in (0, 2)$ correspond precisely to $\mathbb{P}(|a_{ij}| > x) \approx x^{-\alpha}$ with $\alpha \in (0, 2)$. In this text, we fill in the gap of understanding the role of α in the fluctuations linear spectral statistics: when $\alpha \in (2, 4)$, we prove a central limit theorem for $\frac{1}{N}\operatorname{Tr}\varphi(A)$ in the case where φ is a sum of resolvent functions, it appears that the order of the fluctuations, in this case, is $N^{-\alpha/4}$. This completes the picture, summarized in Table 1. Viewed in the light of concentration inequalities for linear spectral functionals of random matrices, random matrices with half-heavy tailed entries interpolate between two extreme regimes, as shown in Table 1:

- Using only the independence of the entries, (see [11, Lem. C.1] or [22,9,10]) for any bounded function $\varphi : \mathbb{R} \to \mathbb{R}$ with finite total variation, we have, for any $\delta > 0$,

$$\mathbb{P}\left(|\operatorname{Tr}\varphi(A) - \operatorname{E}\operatorname{Tr}\varphi(A)| \geq \delta\right) \;\leq\; Ce^{-c\frac{\delta^2}{N\|\varphi\|_{\mathrm{TV}}^2}}, \tag{2}$$

  which proves that

$$\sqrt{N}\left(\frac{1}{N}\operatorname{Tr}\varphi(A) - \operatorname{E}[\frac{1}{N}\operatorname{Tr}\varphi(A)]\right)$$

  is bounded in probability and explains why the order of the fluctuations of (1) cannot be larger than $N^{-1/2}$,
- In the case where the entries of A are independent and satisfy a Log-Sobolev inequality (for example in the GO(U)E case), then for any Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$, we have, by